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2007 Nonlinearity 20 765
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Spectral transition in a sparse model and a class of nonlinear dynamical systems

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Received 21 April 2006, in final form 11 January 2007
Published 9 February 2007
Online at stacks.iop.org/Non/20/765

Recommended by J R Dorfman

Abstract

We introduce a class of Jacobi matrices with sparse potential in the sense that the perturbation of the Laplacian consists of a (direct) sum of fixed off-diagonal $2 \times 2$ matrices placed at sites whose distances from one another grow exponentially. The essential spectrum is proved to be the set $[-2, 2]$. The model may be formulated in terms of angular variables, the Prüfer angles, in terms of which is defined a nonlinear dynamical system with two control parameters, $p$—the ‘impurity strength’—and $\beta$—the ‘sparseness’. A method to study the spectrum of these matrices which exploits the existence, for fixed energy, of a continuous asymptotic distribution function (a.d.f.) of the Prüfer angles is introduced. This is in contrast to the method introduced by Pearson (1978 Commun. Math. Phys. 60 13–36), which exploits uniform distribution in energy. With the help of metric theorems of ergodic theory and ideas of Zlatoš (2004 J. Funct. Anal. 207 216–52), we are able to use this framework in order to establish a singular continuous spectrum on an interval of positive Lebesgue measure, under suitable conditions on $\beta$ and $p$. In the complementary set of parameters there is a (dense) pure point spectrum, and thus a spectral transition takes place, if a continuous a.d.f is assumed. We provide numerical evidence for this assumption.

Mathematics Subject Classification: 47B36, 47A25, 47A35, 37N20, 37E05
1. Introduction and summary

There have been a few results on models with spectral transition, i.e. supporting spectra of different types for a complementary set of parameters. They include the Anderson transition in a Bethe lattice [K], a sort of metal–insulator transition for almost Mathieu operator [J], as well as a transition from a singular continuous (s.c.) to a pure point (p.p.) spectrum in a special random model with sparse potential [Z]. In this paper, we address the possibility of a spectral transition in a deterministic model, a free Jacobi matrix perturbed by a sparse potential, similar to the one treated in [Z].

We consider off-diagonal Jacobi matrices

\[
J = \begin{pmatrix}
0 & p_0 & 0 & 0 & \cdots \\
p_0 & 0 & p_1 & 0 & \cdots \\
0 & p_1 & 0 & p_2 & \cdots \\
0 & 0 & p_2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

(1.1)

with \(p_n = p_n(\kappa), n = 0, 1, 2, \ldots,\) real-valued functions of a parameter \(\kappa\) which plays the role of a ‘coupling constant’. Without loss of generality, \(0 \leq \kappa \leq 1\) and we assume that \(p_n : [0, 1] \rightarrow \mathbb{R}_+\) is a continuous function for each \(j\). Furthermore, as far as the ‘sparseness’ condition is concerned, \((p_n)_{n \geq 0}\) is required to be of the form

\[
p_n = \begin{cases}
1 - r_j & \text{if } n = a_j \in A, \\
1 & \text{if } n \notin A,
\end{cases}
\]

(1.2)

with \(r_j(0) = 0\) and \(r_j(1) = 1,\)

(1.3)

where \(A = \{a_j\}_{j \geq 1}\) is a set of strictly positive integers such that

\[a_j - a_{j-1} \geq 2, \quad j = 2, 3, \ldots\]

(1.4)

and

\[
\lim_{j \rightarrow \infty} \frac{a_{j+1}}{a_j} = \beta > 1,
\]

(1.5)

where \(\beta\) is the ‘sparseness parameter’.

A concrete example satisfying (1.3) is given by

\[
r_j(\kappa) = \frac{\kappa}{1 + (1 - \kappa) \xi_j}
\]

(1.6)

with \(\xi_j \geq 0\). If \(\xi_j \rightarrow \infty\) as \(j \rightarrow \infty\), then

\[
r_j(\kappa) \rightarrow 0,
\]

(1.7)

for all \(\kappa \in (0, 1)\) and \(r_j(1) = \kappa\). If \(\xi_j = 0\) for all \(j\), then

\[
r_j(\kappa) = \kappa.
\]

(1.8)

In both cases (either \(\xi_j \rightarrow \infty\) or \(\xi_j = 0\), as \(\kappa\) varies in the interval \((0, 1)\), \(J\) interpolates continuously two opposite situations: \(J\) has a pure a.c. spectrum at \(\kappa = 0\) and a purely point spectrum at \(\kappa = 1\). Note that, in both cases, \(p_n(1) = 0\) for all \(n \in A\) and \(J\) becomes a direct sum of finite matrices with a dense p.p. spectrum if (1.5) holds. This is Howland’s remark [H], which shows off-diagonal Jacobi matrices as natural examples in which a spectral transition might be expected. Such transitions also occur, however, in the diagonal case (see theorem 6.3 of [Z] for an example in the random case).
Jacobi matrices (1.1) can be written as
\[
(J u)_n = p_n u_{n+1} + p_{n-1} u_{n-1},
\]
(1.9) and with \(p_{-1} = 1\) fixed and boundary condition \(u_{-1} = 0\). A Jacobi matrix \(J_\alpha\) is said to satisfy an \(\alpha\)-boundary condition at \(-1\) if \(u_{-1}\) is given by
\[
u_{-1} \cos \alpha - u_0 \sin \alpha = 0
\]
(1.10) \((J = J_0\) satisfies a Dirichlet 0-boundary condition at \(-1\), see also (2.12)). We shall call the class of Jacobi matrices (1.9) satisfying (1.10) for some \(\alpha \in [0, \pi)\) and (1.2)–(1.5) Howland’s class \(\mathcal{H}\) and the subset of those satisfying (1.7) \(\mathcal{H}_0\). Note that \(\mathcal{H} \setminus \mathcal{H}_0\) include those in which (1.8) holds.

Let \(J_0^0\) be the free Jacobi matrix
\[
J_0^0 = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & \cdots \\
0 & 1 & 0 & 1 & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
The spectrum \(\sigma(J_0^0) = [-2, 2]\) is purely absolutely continuous. If \(J \in \mathcal{H}_0\), \(Q = J - J_0^0\) is a compact perturbation of \(J_0^0\) and thus, by Weyl’s invariance theorem, the essential spectrum of \(J\) remains the same as that of \(J_0^0\), \(\sigma_{\text{ess}}(J) = \sigma(J_0^0) = [-2, 2]\). The spectrum of \(J\) in this subclass is described as follows.

**Theorem 1.1.** Let \(J \in \mathcal{H}_0\) with \(0 < \kappa < 1\). Then \(\sigma(J)\) is purely a.c. if \(\sum_{k=1}^{\infty} |r_k|^2 < \infty\) and purely s.c. if \(\sum_{k=1}^{\infty} |r_k|^2 = \infty\) and the ‘sparseness parameter’ is \(\beta = \infty\).

Theorem 1.1 may be proven along the lines of theorem 1.7 of Kiselev–Last–Simon [KLS] (see also [KR] and references therein). It shows that there is no spectral transition in the subclass \(\mathcal{H}_0\) as \(\kappa\) varies in \((0, 1)\), at least under the sparseness condition (1.5). In order for a transition to occur at a nontrivial value of \(\kappa\), \(J\) has to be such that (1.7) does not hold (i.e. \(r_n \not\to 0\)) and it is likely to be from s.c. to (dense) p.p., with a.c. spectrum just for \(\kappa = 0\).

When \(r_n \not\to 0\), i.e. (1.8), theorem 1.1 is no longer applicable. This is the case we consider in the rest of the paper.

The literature on sparse potentials is extensive, but one of the very basic methods employed to study the spectrum is given by Pearson [P]. The idea is that sparse ‘bumps’ \(a_k\) lead to the ‘independence’ of certain (deterministic) functions which behave as (functions of) a random variable, uniformly distributed (u.d.) over an interval of energy (see the discussion on [P], p 21). For a recent reference along these lines see [BP], in particular section 5.

In this paper, we introduce a different method, in which the emphasis is on u.d. on the space-variable, rather than on the energy. More precisely, we prove in section 4 (theorem 4.4) that if the Prüfer angles \((\theta_k)_{k \geq 1}\), where \(k\) is the space-variable, are uniformly distributed for almost every value of the energy, there are two complementary sets of parameters \((\beta, \rho)\), both of positive Lebesgue measure, in which the spectra are, respectively, s.c. and (dense) p.p. Similar results also hold if a weaker conjecture (conjecture 4.1) is valid (see remark 4.5.3).

In section 5 the perturbation scheme introduced in [Z] is extended to show that, within our framework, one of the assumptions of theorem 4.4 may be verified if \(\beta\) is sufficiently large. This relies strongly on the metric theorems of ergodic theory [KN]. As a corollary, one obtains a s.c. spectrum on a set of positive Lebesgue measure (theorem 5.8). In section 6 we provide compelling numerical evidence of the validity of the conjecture 4.1, again supported by the metric theorems. Section 2 presents very briefly some results on the spectrum, and
section 3 introduces the Prüfer variables. Section 7 is reserved for a conclusion, with a brief statement on open problems.

2. Some notation and a theorem on the spectrum

A bounded Jacobi matrix $J$ is a matrix of a self-adjoint operator $H$ on a separable Hilbert space $\mathcal{H} : J_{ij} = (\varphi_i, H \varphi_j)$, where $\{\varphi_j\}_{j \geq 0}$ is an orthonormal base of $\mathcal{H}$. The spectral measure $\mu = d\rho$ associated with $J$ is given by the 00-element of the resolvent matrix

$$(J - \lambda I)_{00}^{-1} = (\varphi_0, (H - \lambda)^{-1} \varphi_0) = \int_{-\infty}^{\infty} \frac{1}{x - \lambda} d\rho(x). \quad (2.1)$$

We introduce, for $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, a $2 \times 2$ transfer matrix

$$T(n, n - 1; \lambda) = \begin{pmatrix} \frac{\lambda}{p_n} & -p_{n-1} \\ p_n & 0 \end{pmatrix}, \quad (2.2)$$

associated with the $l_2(\mathbb{N})$-solutions $u$ of $(J - \lambda I)u = 0$ and set

$$T_0(\lambda) = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}, \quad (2.4)$$

for $n = a_j + 1$, for some $j \geq 1$, and for $n = a_j$, for some $j \geq 1$,

$$T(n, n - 1; \lambda) = T_+(a_j; \lambda) := \begin{pmatrix} \frac{\lambda}{1 - r_j} & -\frac{\lambda}{1 - r_j} \\ 1 & 0 \end{pmatrix}. \quad (2.6)$$

Let $\lambda = 2 \cos \varphi$ where $\varphi \in (0, \pi)$ (see theorem 2.1 for a proof that this parametrization is exhaustive for the essential spectrum which will be our only concern) and

$$U = \begin{pmatrix} 0 & \sin \varphi \\ 1 & -\cos \varphi \end{pmatrix}. \quad (2.7)$$

be the matrix $U$ that makes $T_0(\lambda)$ similar to a pure rotation

$$R(\varphi) = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} = UT_0(\lambda)U^{-1}. \quad (2.8)$$

It will be most convenient to introduce, for each $a_j \in A$, the matrices $P_+(a_j)$ given by

$$T_+(a_j) = R(\varphi)P_+(a_j)R(\varphi), \quad (2.9)$$

where $T_+(a_j) = U^T(a_j)T_-(a_j)U^{-1}$. From (2.3) to (2.9) and, since a product of rotations is a rotation, we have

$$UT(n; \lambda)U^{-1} = R((n - a_k)\varphi)P_+(a_k)R((a_k - a_{k-1})\varphi) \cdots P_+(a_1)R((a_1)\varphi), \quad (2.10)$$

which will be useful in section 3.

According to (1.2) and (1.8), in the above equations

$$r_j = 1 - p_{a_j}, \quad (2.11)$$

where $(p_{a_j})_{j \geq 1} \equiv 0$ refers to the p.p. case and $(p_{a_j})_{j \geq 1} \equiv 1$ to the unperturbed (a.c.) case.
Theorem 2.1. If \( J \in \mathcal{H} \) and \( 0 \leq r_j \leq 1 \) for all \( j \in \mathbb{N} \), i.e. \( p_{aj} \in [0, 1] \), then
\[
\sigma_{\text{ess}}(J) = [-2, 2]
\]
is the essential spectrum of \( J \).

Proof. With \( J_0 \) given by (1.9) with the 0-boundary condition at \(-1\), we have
\[
J_\alpha = J_0 + \tan \alpha \delta_0.
\]
Clearly, \((\delta_0 u)_n = \delta_{n0} u_0\) is a rank-one perturbation of \( J_0 \) and \( \sigma_{\text{ess}}(J_\alpha) = \sigma_{\text{ess}}(J_0) \), by Weyl’s invariance principle. Thus, the choice \( J = J_0 \) entails no loss of generality, as far as the essential spectrum is concerned.

We prove that \( \sigma(J_0) = [-2, 2] \). Thus, there is no discrete spectrum and \( \sigma(J_0) = \sigma_{\text{ess}}(J_\alpha) \) for any \( \alpha \in [0, \pi) \).

Let \( J_0 \in \mathcal{H} \) with \((p_n)_{n \geq 0} \) so that \( 0 \leq p_{aj} \leq 1 \), \( \forall j \). To prove \( \sigma(J_0) \subseteq [-2, 2] \), let \( u = (u_n)_{n \geq 0} \in l^2(\mathbb{N}) \). We have
\[
(u, J_0 u) = \sum_{n=0}^{\infty} (u_n u_{n+1}) h_n \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix},
\]
where
\[
h_n = \begin{pmatrix} 0 & p_n \\ p_n & 0 \end{pmatrix}.
\]
Inserting
\[
h_n = \lambda_n^+ p_n^+ + \lambda_n^- p_n^-
\]
into (2.13), where \( p_n^\pm \) projects onto the eigenvector associated with the eigenvalue \( \lambda_n^\pm = \pm p_n \), we have
\[-2 \tilde{\lambda} \leq (u, J_0 u) \leq 2 \tilde{\lambda},
\]
where \( \tilde{\lambda} = \sup_n \lambda_n^+ = \sup_n p_n = 1 \). Hence
\[
\sigma(J_0) \subseteq [-2 \tilde{\lambda}, 2 \tilde{\lambda}] = [-2, 2].
\]
Note that, because \( J_0 \) is a symmetric off-diagonal matrix, the spectrum is symmetric with respect to the origin. In order to prove the opposite inclusion
\[
\sigma(J_0) \supseteq [-2, 2],
\]
we use Weyl’s criterion (see theorem VII.12 of [S1]): if \( A \) is a bounded self-adjoint operator in separable Hilbert space \( \mathcal{H} \), \( \lambda \) is a point of the spectrum \( \sigma(A) \) of \( A \) if and only if there exists a sequence \((\psi_n)_{n \geq 0} \) in \( \mathcal{H} \), with \( \|\psi_n\| = 1 \), such that
\[
\lim_{n \to \infty} \| (A - \lambda) \psi_n \| = 0.
\]
Let \( \lambda = 2 \cos \varphi, \varphi \in [0, \pi] \) and define \( \psi_n = (1/\sqrt{n})(e^{i\varphi}, e^{i2\varphi}, \ldots, e^{in\varphi}, 0, \ldots) \). Clearly \( \psi_n \in l^2(\mathbb{N}) \) and we claim that
\[
\| (J_0 - \lambda) \psi_n \| \leq c \frac{\ln n}{\sqrt{n}}
\]
holds with \( c = c(\beta) \) independent of \( n \) for any \( J_0 \in \mathcal{H} \). In order to prove (2.17), note that \( (J_0 - \lambda) \psi_n \) consists of the action on \( \psi_n \) of a sum of local matrices, each bounded in norm.

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4 Although \( J_\alpha \) may have some discrete spectrum, it will not concern us in what follows.
by one. Two of them involve the extremes $e^{i\phi}$ and $e^{i\eta}$, and there are $O(\ln n)$ nondiagonal matrices, each acting just on two successive components of $\psi_n$. The $O(\ln n)$ comes from the sequence $(a_j)_{j \geq 1}$ satisfying the sparseness condition (1.5), $a_j \sim \beta^j$, with at most $r$ points $a_j$ within $[1, n]$, where $r$ is such that $\beta^r \leq n$ or, equivalently,

$$r \leq \frac{\ln n}{\ln \beta}$$

This proves (2.17) and therefore (2.15).

Henceforth $\lambda \in [-2, 2]$ will be a point of $\sigma_{\text{ess}}(J)$.

3. Prüfer variables

For $\lambda \in \mathbb{C}$ and $n \in \mathbb{N}$, let $T(n, n - 1; \lambda)$ be the $2 \times 2$ transfer matrix given by (2.2) and note that if $u_n = \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$ is a vector in $\mathbb{C}^2$ composed of the components of a $l_2(\mathbb{N})$-solution $u = (u_0, u_1, \ldots)$ of $(J - \lambda)u = 0$, then

$$u_{n+1} = T(n, n - 1; \lambda) \ u_n$$

(3.1) holds for $n = 0, 1, \ldots$, with an initial condition $u_0 = \begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}$ satisfying (1.10) for some $\alpha \in [0, \pi]$.

We use equation (2.10) together with (3.1) and (2.3) to define the Prüfer variables $(R_k, \theta_k)_{k \geq 0}$. For this, we need an expression for $P_{+}^{-}(a)$ at each value $a \in A = \{a_k\}_{k \geq 1}$. Inserting

$$R(-\psi)UT(a)U^{-1} = \begin{pmatrix} p_a & 0 \\ (1 - p_a) \cot \psi & 1 \end{pmatrix}$$

and

$$UT(a)U^{-1}R(-\psi) = \begin{pmatrix} 1 & 0 \\ (1/p_a - 1) \cot \psi & 1/p_a \end{pmatrix}$$

into (2.9), where $T(a), U$ and $R$ are given by (2.5)–(2.8), yields

$$P_{+}^{-}(a) = \begin{pmatrix} p_a & 0 \\ qa \cot \psi & 1/p_a \end{pmatrix}$$

with $qa = 1/p_a - p_a$. If

$$v_k = (R_k \cos \theta_k, R_k \sin \theta_k)$$

(3.3) and

$$\tilde{v}_k = \begin{pmatrix} R_k \cos \tilde{\theta}_k, R_k \sin \tilde{\theta}_k \end{pmatrix}$$

(3.4)

are defined by

$$v_k = R((a_k - a_{k-1})\psi) \ v_{k-1}$$

(3.5) and

$$\tilde{v}_k = P_{+}^{-}(a_k) \ v_k$$

(3.6)

with $a_0 = 0$ and the initial condition

$$\tilde{v}_0 = \begin{pmatrix} \cos \theta_0 \\ \sin \theta_0 \end{pmatrix} = U \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix} = \begin{pmatrix} \sin \alpha \sin \psi \\ \cos \alpha - \sin \alpha \cos \psi \end{pmatrix}$$

(3.7)
then, we have
\[ U^k u_k = \tilde{v}_k = v_{k-1}. \]
Since \( \partial \theta_0 / \partial a = -\sin \varphi \), for each \( \varphi \in (0, \pi) \) there is a one-to-one correspondence between the initial Prüfer angle \( \theta_0 \) and the \( a \)-boundary condition at \(-1\) of (1.9). It follows that the Euclidean norm of \( UT (n; \lambda) U^{-1} \), given by
\[
\| UT (n; \lambda) U^{-1} v_0 \|^2 = R^2_N,
\]
for any \( n \) such that \( a_N \leq n < a_{N+1} \), can thus be written as
\[
R^2_N = \frac{R^2_N}{R^2_{N-1}} \cdots \frac{R^2_2}{R^2_1} \frac{1}{N} \sum_{k=1}^{N} \left( f \left( p_{ak}, \theta_k \right) - \ln p_{ak}^2 \right) \right)^N,
\]
where
\[
f \left( p, \theta \right) = \ln \left( p^4 \cos^2 \theta + \left( \sin \theta + \left( 1 - p^2 \right) \cot \varphi \cos \theta \right)^2 \right)
\]
with the Prüfer angles \( (\theta_k)_{k \geq 0} \) satisfying the recursion relation
\[
\theta_{k+1} = g \left( p_{ak}, \theta_k \right) + \left( a_{k+1} - a_k \right) \varphi,
\]
where
\[
g \left( p, \theta \right) = \tan^{-1} \left( \frac{1}{p^2} \left( \tan \theta + \cot \varphi \right) - \cot \varphi \right).
\]
for \( k \geq 1 \), with \( \theta_1 \) given by
\[
\theta_1 = \theta_0 + a_1 \varphi.
\]

4. The main theorem

In this section we prove the main theorem (theorem 4.4) connecting continuous asymptotic distribution of the Prüfer angles with the spectral transition. We combine proposition 4.3 stated below with the powerful theorems of Last and Simon [LS] relating the nature of the spectrum of \( J \) to the properties of the transfer matrix \( T(n; \lambda) \): the essential support \( \Sigma_{ac} \) of the a.c. part \( \mu_{ac} \) of spectral measure \( \mu = d\rho \) (2.1) is given by
\[
\Sigma_{ac} = \left\{ \lambda : \lim inf_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \| T (n; \lambda) \|^2 < \infty \right\}.
\]

By theorems 1.6 and 1.7 of [LS] the eigenvalue equation \((J - \lambda I)u = 0\) has no \( l_2(\mathbb{N})\)-solutions if
\[
\sum_{n=0}^{\infty} \| T (n; \lambda) \|^{-2} = \infty.
\]
(see also theorem 2.1 in [SS]) and an \( l_2(\mathbb{N})\)-solution if
\[
N_{pp} \equiv \sum_{n=0}^{\infty} \| T (n; \lambda) \|^2 \left( \sum_{k=0}^{n} \| T (k; \lambda) \|^{-2} \right)^2 < \infty.
\]
We say that the Prüfer angles $\theta_k \geq 1$ defined by (3.3) have an asymptotic distribution function modulo $\pi$ (abbreviated a.d.f. mod $\pi$) $\nu(x)$ if, for any $\theta \in [0, \pi]$, 
\[
\lim_{N \to \infty} \frac{\text{card} \left\{ k : \theta_k \in [0, \theta), 1 \leq k \leq N \right\}}{N} = \nu(\theta), \tag{4.4}
\]
where $\text{card}(S)$, the cardinality of the set $S$, is the number of elements in $S$. Note that the function $\nu : [0, \pi] \to [0, 1]$ is nondecreasing right continuous with $\nu(0) = 0$ and $\nu(\pi) = 1$.

The sequence $(\theta_k)_{k \geq 1}$ need not have an a.d.f. mod $\pi$ but upper
\[
\limsup_{N \to \infty} \frac{\text{card} \left\{ k : \theta_k \in [0, \theta), 1 \leq k \leq N \right\}}{N} = \Phi_1(\theta) + (\theta) \tag{4.5}
\]
and lower
\[
\liminf_{N \to \infty} \frac{\text{card} \left\{ k : \theta_k \in [0, \theta), 1 \leq k \leq N \right\}}{N} = \Phi_1(\theta) - (\theta) \tag{4.6}
\]
distribution functions, with $0 \leq \Phi_1(\theta) \leq \Phi_1(\theta) \leq 1$ for $\theta \in [0, \pi]$, always exist. If $\Phi_1(\theta) = \Phi_1(\theta)$, the sequence $(\theta_k)_{k \geq 1}$ has an a.d.f. mod $\pi$.

For simplicity, and without loss of generality, we fix here and henceforth $p_k \equiv p$ for all $k$, $0 < p < 1$, and take $A = (a_k)_{k \geq 1}$ so that
\[
a_k - a_{k-1} = \beta^k \tag{4.5}
\]
with $a_0 = 0$, i.e. $a_k = a_k(\beta) = (\beta^{k+1} - \beta)/(\beta - 1)$, $\beta > 1$. In this case (3.9) and (3.11) read
\[
R_N^2 = \left( \frac{1}{p^2} \exp \left( \frac{1}{N} \sum_{k=1}^{N} f(\theta_k) \right) \right)^N \tag{4.6}
\]
with
\[
\theta_k = g(\theta_{k-1}) + \beta^k \psi \tag{4.7}
\]
where $f(\theta) = f(p, \theta)$ is given by (3.10), $g(\theta) = g(p, \theta)$ by (3.12) and $\theta_1$ by (3.13).

**Conjecture 4.1.** The sequence $(\theta_k)_{k \geq 1}$ has the continuous a.d.f. mod $\pi$, $\nu(\theta)$, for $\phi \in [0, \pi]$ \setminus $A_0$, where $A_0$ is a set of Lebesgue measure zero, possibly $\theta_0$-dependent.

We have (see theorem 7.2, chapter 1 of [KN]) the following theorem.

**Theorem 4.2.** A sequence $\omega = (\theta_k)_{k \geq 1}$ of real numbers has the continuous a.d.f. mod $\pi$, $\nu(\theta)$, if and only if
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(\theta_k) = \frac{1}{\pi} \int_{0}^{\pi} f(\theta) \, d\nu(\theta) \tag{4.8}
\]
holds for every continuous periodic function $f$ of period $\pi$.

The other ingredient is the following result.

**Proposition 4.3.** Suppose
\[
[a, b] \subset \sigma (J_0) \tag{4.9}
\]
holds for all $\alpha \in [0, \pi]$ and there exists $A \subset [a, b]$ of Lebesgue measure zero independent of $\alpha$ such that, if $\lambda \in [a, b] \setminus A$,
\[
(J - \lambda I) u = 0 \tag{4.10}
\]
has a solution $u \in l_2(\mathbb{N})$, where $J$ is given by (1.9) for $n = 1, 2, \ldots$, and $u_0$ may take any value. Then $\sigma (J_0)$ has only dense p.p. spectrum in $[a, b]$ for a.e. (w.r.t. Lebesgue measure) $\alpha$-boundary condition (1.10).
Proof. Take \( \lambda \in [a, b] \backslash A \). By (4.10), (1.9) has, for this \( \lambda \), a \( l_2 \)-solution on \((0, \infty)\). Thus, by theorem I I.3 of [S2], either (1) \( \lambda \) is an eigenvalue of \( J \) with Neumann b. c. (i.e. corresponding to \( u_{-1} = 0 \) in (1.10) or \( \alpha = 0 \)) or (2) \( G(\lambda) = \int d\rho(y)/(\lambda - y)^2 < \infty \), where \( \rho \) is the spectral measure (2.1). The first alternative (1) can hold at most for \( \lambda \) belonging to a countable set, but, by hypothesis, (4.10) holds in a set of positive Lebesgue measure, i.e. the complement of a set of zero Lebesgue measure. Thus, even excluding the countable set satisfying the first alternative, we are led to the statement that \( G(\lambda) < \infty \) for a.e. \( \lambda \) in \([a, b]\) and therefore, by the Simon–Wolff criterion [SW] (see also theorem II.5 of [S2]), \( J_a \) has only p.p. spectrum in \([a, b]\) for a.e. boundary condition \( \alpha \), which is a nontrivial statement by assumption (4.9). Finally, the inclusion of those \( \lambda \) satisfying the first alternative does not alter the statement of the proposition. \( \square \)

The assumption of the following theorem—our main result—holds for certain limit values of parameters \( p, \beta \) and \( \lambda = 2 \cos \varphi \), but the scenery described differs only slightly in their neighbourhood (see section 6 for compelling numerical evidence and theorem 5.8 for a rigorous result).

Theorem 4.4. Let \( J_a \in H \backslash \mathcal{H}_0 \) with \( r_j = 1 - p \) for all \( j \in \mathbb{N} \), \( p \in (0, 1) \) and \( \beta \in \mathbb{N}, \beta \geq 2 \). Let

\[
I_1 \equiv \left\{ \lambda \in [-2, 2] : \left( \frac{p}{1 - p^2} \right)^2 (\beta - 1) (4 - \lambda^2) > 1 \right\}
\]

and

\[
I_2 \equiv \left\{ \lambda \in [-2, 2] : \left( \frac{p}{1 - p^2} \right)^2 (\beta^2 - 1) (4 - \lambda^2) < 1 \right\}.
\]

If the Prüfer angles \((\theta_k)_{k \geq 1}\) are uniformly distributed modulo \( \pi \) for \( \varphi \in [0, \pi] \backslash A_{\theta_0} \), where \( A_{\theta_0} \) is a set of Lebesgue measure zero, possibly \( \theta_0 \)-dependent, then

(a) there exists a set \( A_1 \) of Lebesgue measure zero such that the spectrum restricted to the set \( I_1 \backslash A_1 \) is purely singular continuous,

(b) the spectrum \( \sigma(J_a) \) of \( J_a \) is a dense pure point when restricted to \( I_2 \) for almost every \( \alpha \in [0, \pi) \), where \( \alpha \) characterizes the boundary condition (1.10).

Remark 4.5.

1. Take, e.g., \( \beta = 2 \): (4.11) holds for \( p \) sufficiently close to 1, i.e. for small ‘coupling’ \( \kappa = 1 - p \), and \( \lambda \) never in a neighbourhood \( U \) near the edges of the band \( \pm 2 \), while (4.12) holds for \( p \) sufficiently close to zero (‘large’ \( \kappa \)) and \( \lambda \) always close to the band edges (see figure 1). These features are also expected in the Anderson model, with the continuous part of the spectrum being, however, absolutely continuous.

2. In section 6 we present numerical evidence that the sequence \((\theta_k)_{k \geq 1}\) has a continuous a.d.f. mod \( \pi \) for any parameter values \( p \in (0, 1) \) and \( \beta = 2 \) and becomes u.d. mod \( \pi \) in the limit as \( p \) goes to both 0 and 1.

3. The proof of theorem 4.4 holds even when the Prüfer angles are not nearly uniformly distributed under the assumption that the corresponding lhs of the integral (4.17), with \( dv(\theta) \) replacing \( d\theta \), is close to the rhs of (4.17). If \((\theta_k)_{k \geq 1}\) has a continuous a.d.f. mod \( \pi \), the rhs of (4.8) with \( f \) given by (4.14) can be evaluated numerically in order to determine intervals \( I_1 \) and \( I_2 \) similarly to (4.11) and (4.12). Figures 2 and 3 show the profile of the integral as a function of \( \varphi \) for \( p = 0.1 \) and \( p = 0.3 \). The dots represent the numerical integration of (4.17) with the Lebesgue measure \( d\theta \) replaced by the Stieltjes–Lebesgue measure \( dv \) where \( v \) is the corresponding distribution given in figure 4.
4. Theorem 4.4 holds without the uniform distribution condition if $A = (a_j)_{j \geq 1}$ in (1.2) is given by a sequence of independent random variables, defined in a probability space $(\Omega, \nu)$, with $a_j = a_j(\omega)$ uniformly distributed in $\{\beta^j - j, \ldots, \beta^j + j\}$. It follows from section 6 of [Z] that the lhs of (4.13) converges to 0 for almost every realization $A(\omega)$ and the inequalities in $I_1$ and $I_2$ may be sharpened as in theorem 6.3 of [Z].

Proof. We must verify that the conditions on $p$, $\beta$ and $\lambda = 2 \cos \varphi$ which define (4.11) and (4.12) imply (4.2) and (4.3), respectively, for some $J = J_\alpha$ satisfying the $\alpha$-boundary condition at $-1$.

By assumption (4.8) the large $N$-behaviour of $R_N$ is governed by (4.6) with the sum in the exponent substituted by the integral (4.8). More precisely, Koksma’s inequality gives (see theorem 5.1 and corollary 1.1 in chapter 2 of [KN])\(^5\)

\[
\left| \frac{1}{N} \sum_{k=1}^{N} f(\theta_k) - \frac{1}{\pi} \int_{0}^{\pi} f(\theta) \, d\theta \right| \leq \frac{D_N}{\pi} \int_{0}^{\pi} |f'(\theta)| \, d\theta, \tag{4.13}
\]

\(^5\) The Lebesgue measure $d\theta$ may be substituted by the Riemann–Stieltjes measure $d\nu(\theta)$ in this theorem.
Figure 3. Profile of the integral (4.17) as a function of $\phi$ for $p = 0.3$.

Figure 4. Histograms with $10^7$ iteration points of the map (4.7) for $\beta = 2$, $\psi$ a dyadic irrational and $p^2 = 0.01, 0.1, 0.3, 0.7, 0.9$ and 0.99, respectively.

(This figure is in colour only in the electronic version)

where the discrepancy

$$D_N^* := \sup_{0 < \alpha \leq \pi} \left| \frac{\text{card}(\{k : \theta_k \in [0, \alpha), 1 \leq k \leq N\})}{N} - \alpha \right|$$

goese to 0 as $N$ goes to $\infty$. 
In order to employ theorem 4.2 it is necessary to ascertain that $f$ is a Riemann-integrable periodic function of period $\pi$ and compute its integral. By simple trigonometric identities (3.10) may be written in the form
\[ f(\theta) = \ln(a + b \cos 2\theta + c \sin 2\theta), \] (4.14)
where
\[ 2a = (1 - p^2)^2 \cot^2 \varphi + 1 + p^4, \]
\[ 2b = (1 - p^2)^2 \cot^2 \varphi - 1 + p^4 \]
and
\[ c = (1 - p^2) \cot \varphi, \]
which shows that $f$ is periodic with period $\pi$. An explicit computation gives
\[ a + b \cos 2\theta + c \sin 2\theta \geq \min_{\theta} (a + b \cos 2\theta + c \sin 2\theta) \]
\[ = a - \sqrt{b^2 + c^2} = a - \sqrt{a^2 - p^4} > 0, \] (4.15)

since
\[ 0 < b^2 + c^2 = a^2 - p^4 \] (4.16)
if $0 < p < 1$, and this implies that $f(\theta)$ is Riemann-integrable. From [PBM, p 546], (4.14) and (4.16), we have
\[ \frac{1}{\pi} \int_0^\pi f(\theta) d\theta = \ln \left( \frac{a + \sqrt{a^2 - b^2 - c^2}}{2} \right) = \ln \left( \frac{(1 - p^2) \csc^2 \varphi}{4} + p^2 \right). \] (4.17)

Now let,
\[ r(p, \varphi) \equiv 1 + \frac{(1 - p^2) \csc^2 \varphi}{4p^2}. \] (4.18)

By (4.6), (4.13), (4.14) and (4.17),
\[ C_N^{-1} r_N \leq R_N^2 \leq C_N r_N, \]
for $\varphi \in [0, \pi] \setminus A_{\theta_0}$, (4.19)
where $A_{\theta_0}$ is a set of Lebesgue measure 0, possibly depending on $\theta_0$, mentioned in the hypothesis, and some finite constant $C_N > 1$ such that $\lim_{N \to \infty} C_N^{1/N} = 1$. Note that $\sup_{\varphi \in [0,\pi]} |f'(\varphi)| < \infty$ in view of (4.15), so that the rhs of (4.13) indeed yields a $C_N$ with the stated properties.

Let $\|A\|_U := \|UAU^{-1}\|_U$ be a matrix norm defined by the invertible matrix $U$ given by (2.7) with $\varphi \in (0, \pi)$, where
\[ \|B\| = \sup_{v \in \mathbb{C}^2} \frac{\|Bv\|}{\|v\|}. \]
Let $n$ be such that $a_N \leq n < a_{N+1}$ holds for some $N \in \mathbb{N}$. We have
\[ \sum_{m \geq n} \|T (m; \lambda)\|_{B^2}^2 = \sum_{k \geq N+1} \|T (a_k; \lambda)\|_{B^{k+1}}^2 + \|T (a_N; \lambda)\|_{B^2}^2 \sum_{n \leq m < a_{N+1}} 1. \] (4.20)

By the method in [KLS, lemma 2.2 and theorem 2.3], it can be proved that
\[ \|T (m; \lambda)\|_{B^2}^2 \leq C^2 \|T (a_k; \lambda)\|_{B^2}^2 \leq C^2 \left( \max_{|\epsilon| \leq 1} R_k(\theta_0^\epsilon) \right)^{-2}. \] (4.21a)
\[ \| T (m; \lambda) \|^{-2} \geq C^{-2} \| T (a_k; \lambda) \|^{-2} \geq \tilde{C}^{-2} \left( \max_{\theta \in [1,2]} R_k(\theta) \right)^{-2} \]  
(4.22a)

hold with \( C = \sqrt{(1 + |\cos \varphi|) / (1 - |\cos \varphi|)} \) and \( \tilde{C} = C / |\sin (\theta_0^1 - \theta_0^2) / 2| \) for any pair \( \theta_0^1, \theta_0^2 \in [0, \pi] \) with \( 0 < |\theta_0^1 - \theta_0^2| < \pi / 2 \). Now let,

\[ S_{N,M}^\pm = \beta \sum_{k=N}^{M} \sum_{m=0}^{\#M} C^{p - 1} \left( \frac{\beta}{r} \right)^k \]  
(4.23)

By (4.19)–(4.23),

\[ \tilde{C}^{-2} S_{N+1,M}^- \leq \sum_{m=m}^{\#M} \| T (m; \lambda) \|^{-2} \leq C^2 S_{N,M}^+ \]  
(4.24)

holds for all integers \( M \geq N + 1 \) and \( \lambda \in B \) where

\[ B = 2 \cos \left( [0, \pi] \setminus A' \right) \]  
(4.25a)

and

\[ A' = A_{[0]} \cup A_{[1]} \]  
(4.26a)

is a fixed set of Lebesgue measure zero, once we have chosen an arbitrary but from now on fixed set \( \{ \theta_0^1, \theta_0^2 \} \). The further assertions of the lemma depend solely on the bounds (4.24).

By (4.18), (4.19) and (4.25a)

\[ B \subseteq \left\{ \lambda : \lim_{n \to \infty} \| T (n; \lambda) \| = \infty \right\} \]  
(4.27)

and lies in the complement of \( \Sigma_{ac} \) by (4.1). Thus, except for a set

\[ A_1 = A' \cup A'' \]  
(4.28)

of Lebesgue measure zero—where \( A'' \) is a set of zero Lebesgue measure connected with the definition of the essential support \( \Sigma_{ac} \) of \( \mu_{ac} \) (see, e.g., [CFKS, p 179], for this definition)—the set

\[ B_1 = 2 \cos \left( [0, \pi] \setminus A_1 \right) \]  
(4.29)

lies in the singular spectrum. By (4.23) and the lhs of (4.24), (4.2) holds (taking \( M \to \infty \)) if

\[ 4p^2(\beta - 1) > (1 - p^2)^2 \csc^2 \varphi, \]  
(4.30)

which, using \( 4 \csc^2 \varphi = 4 - \lambda^2 \), may be written in the form (4.11) and defines \( I_1 \). Thus, by (4.29) the spectrum is purely singular continuous when restricted to the set

\[ I_1 \setminus (A_1 \cap I_1) \],

which is assertion (a) of theorem 4.4.

We now prove assertion (b), assuming that

\[ 4p^2(\beta - 1) < (1 - p^2)^2 \csc^2 \varphi, \]  
(4.31)

which, by (4.18) and (4.23), implies that \( S_{N,M}^+ \) converges for each \( N \) as \( M \to \infty \). Using again (4.24) and denoting by

\[ S_N^+ = \lim_{M \to \infty} S_{N,M}^+, \]  
(4.32)

6 Here, \( f (A) \) denotes the set \( \{ f (x) : x \in A \} \).
we find
\[ \sum_{m=n}^{\infty} \| T(m; \lambda) \|^{-2} \leq C^2 S_N. \] (4.33)

By (4.19), (4.21a), (4.32) and (4.33), and recalling the notation in (4.3), there exist finite constants \( C' \) and \( C'' \) such that
\[ N_{pp} \leq C' \sum_{N=1}^{\infty} \beta^N R_N^2 (S_N^2)^2 \leq C'' \sum_{N=1}^{\infty} C_N^3 \left( \frac{4 p^2 \beta^3}{(1 - p^2)^2 \csc^2 \varphi + 4 p^2} \right)^N. \]

The above series converges if
\[ \frac{p^2}{(1 - p^2)^2} (\beta^3 - 1)(4 - \lambda^2) < 1, \] (4.34)
which defines \( I_2 \) in (4.12). Thus, by (4.3) and (4.29) if \( \lambda \) belongs to the set \( B_2 = I_2 \setminus (A_1 \cap I_2) \)
the eigenvalue equation \( (J - \lambda I) u = 0 \) has an \( l^2 \) \((N)\)-solution. Since
\[ I_2 \subset \sigma (J_\alpha), \quad \forall \alpha \in [0, \pi), \]
by theorem 2.1, the assumptions of proposition 4.3 hold and \( J_\alpha \) has only (dense) p.p. spectrum in \( I_2 \) for almost every \( \alpha \in [0, \pi) \), concluding the proof of assertion (b) of theorem 4.4. \( \square \)

Remark 4.6. It follows from a general result of rank-one perturbations (see, e.g., theorem II.11 in [S2]) that, for a dense \( G_\delta \) set of \( \alpha \)-boundary conditions at \(-1\), \( J_\alpha \) has a purely s.c. spectrum on \( I_2 \). The \( G_\delta \) set in \([0, \pi]\) has, by proposition 4.3, Lebesgue measure zero.

5. Uniform distribution of Prüfer angles

In this section the conclusions of theorem 4.4 will be extended beyond its assumption on uniform distribution of Prüfer angles by applying the perturbation scheme of [Z] to the problem at hand. The main result is theorem 5.8. We observe that the proof of theorem 4.4 remains unaltered if the Prüfer angles \( \{ \theta_k \} \) are replaced by an uniformly distributed sequence \( \{ \xi_k \} \). By (4.6), (4.13) and (4.19), we need only evaluate the error
\[ E_N = \frac{1}{N} \sum_{k=1}^{N} (f(\theta_k) - f(\xi_k)) \] (5.1)
of replacing Birkhoff averages associated with \( \{ \theta_k \} \) by one with \( \{ \xi_k \} \) and show that it can be made so small that, with a slight modification in the spectral conditions (4.11) and (4.12), the conclusions of theorem 4.4 hold.

Given a sequence of numbers \( \{ \xi_k \} \) let \( (S_n)_{n \geq 1} \) be given by
\[ S_n = \frac{1}{n} \sum_{k=1}^{n} e^{2i\xi_k}. \] (5.2)

According to the Weyl criterion [KN, theorem 2.1 of chapter 1], the sequence \( \{ \xi_k \} \) is u.d. mod \( \pi \) if and only if \( \lim_{n \to \infty} S_n = 0 \) for every integer \( \lambda \neq 0 \). If \( \xi_k = \xi_k(x) \) is defined for each \( x \) lying in some interval \([a, b] \subset \mathbb{R} \), then \( S_n = S_n(x) \) is also defined for each \( x \in [a, b] \).

Denoting by \( \| f \|_{L^2([a, b])} = \left( \int_a^b |f(x)|^2 \, dx \right)^{1/2} \) the \( L^2 \) norm, the general metric result is as follows (see theorem 4.2, chapter 1, of [KN] for a proof).
Theorem 5.1. If the numerical series
\[ \sum_{n=1}^{\infty} \frac{1}{n} \| S_n \|_{L^2([a,b])} \]
converges for each integer \( h \neq 0 \) then the sequence \( \{ \xi_h \} \) is u.d. \( \mod \pi \) for almost all \( x \in [a, b] \), i.e. \( \{ \xi_h \} \) is u.d. \( \mod \pi \) for every \( x \) lying in \( [a, b] \) apart from a set with Lebesgue measure 0.

Now, let \( \{ \xi_n \}_{n \geq 1} \) be a sequence of continuous piecewise linear functions \( \xi_n = \xi_n(\phi) \) of \( \phi \in [0, \pi] \) which coincide with \( \theta_n = \theta_n(\phi) \) on every value \( \phi_{n,j} \) at which \( \theta_n(\phi_{n,j}) \) is a multiple of \( \pi \). Zlatos (see appendix A of [Z]) has introduced this sequence to estimate the Hausdorff dimension of invariant subsets of \([0, \pi] \) for which the Birkhoff averages do not exist. We show here that the general metric theorem can be applied to it. For this, we need some properties of (3.12).

Proposition 5.2. The function \( g \) given by (3.12) maps \([ -\pi/2, \pi/2 ] \) onto itself and has three fixed point solutions: \( \theta^s = \pm \frac{\pi}{2} \) are two stable solutions and \( \theta^u = \phi - \frac{\pi}{2}, \phi \in (0, \pi) \), is unstable. \( g \) is monotone increasing with \( g'(\theta^s) = p^2, g'(\theta^u) = 1/p^2 \) and there are two inflection points \( \theta^{\pm} \) such that
\[ g'(\theta^{\pm}) = c_1 p^2 \leq g'(\theta) \leq c_2 p^2 = g'(\theta^s) \]
holds for \( c_1 = c_i(p, \phi) = 1 + O(p^2), i = 1, 2 \).

Proof. Writing \( g(\theta) = \tan^{-1} \left( \frac{\zeta}{p^2} \right) \)
where
\[ \zeta(\theta) = \tan \theta + (1 - p^2) \cot \phi, \]
it follows from
\[ g'(\theta) = \frac{p^2}{p^4 + \zeta^2 \sec^2 \theta} \]
that \( g \) is a monotone increasing function with \( g(-\pi/2) = -\pi/2 \) and \( g(\pi/2) = \pi/2 \). Hence \( g : [ -\pi/2, \pi/2 ] \rightarrow [ -\pi/2, \pi/2 ] \). As \( \theta \) goes to \( \pm \pi/2, \sec^2 \theta / \zeta^2 \rightarrow 1 \) and \( g'(\theta) \rightarrow p^2 \).
Since \( p < 1, \theta = \pm \pi/2 \) are stable fixed points of \( g \). To find the unstable fixed point we observe that
\[ \tan g(\theta) + \cot \phi = \frac{1}{p^2} (\tan \theta + \cot \phi), \]
together with the fixed point equation \( g(\theta) = \theta \) and \( p < 1 \), implies
\[ \tan \theta^u + \cot \phi = 0, \]
which is equivalent to \( \theta^u = \phi - \pi/2 \). At \( \theta = \theta^u \) equation (5.4) reads \( \zeta(\theta^u) = p^2 \cot \phi \) which together with \( \sec^2 \theta^u = 1 + \tan^2 \theta^u = 1 + \cot^2 \phi \) and (5.5) gives
\[ g'(\theta^u) = \frac{1}{p^2} \]
A direct computation yields
\[ g''(\theta) = \frac{2p^2 \sec^2 \theta}{(p^4 + \zeta^2)^2} \left[ (p^4 + \zeta^2) \tan \theta - \zeta \sec^2 \theta \right], \]
from where we get the two inflection points by solving \( (p^4 + \zeta^2) \tan \theta - \zeta \sec^2 \theta = 0 \) for \( \tan \theta \):
\[ \theta^{i \pm} = \tan^{-1} \left( \mp (\cot \phi)^{\pm 1} \left( 1 \pm \frac{p^2}{\sec^2 \phi} + O(p^4) \right) \right) \]
gives $\theta'_i = \theta'' + O(p^2)$ and $\theta'' = \varphi + O(p^2)$, if $\varphi \in [0, \pi/2)$, and $\theta'_{\varphi} = \varphi - \pi + O(p^2)$ if $\varphi \in [\pi/2, \pi)$. Equation (5.3) follows by perturbation around $p = 0$ and this concludes the proof of proposition 5.2. \hfill \Box

If $p^2 \beta > c_2$, then (4.7) and (5.3) imply
\[
\theta'_i(\varphi) = g'(\theta_{i-1})\theta'_{k-1}(\varphi) + \beta^k \leq \frac{c_2}{p^2} \theta'_{k-1} + \beta^k \leq \frac{1}{1 - c_2/(p^2 \beta)} \beta^k,
\]
by induction. Analogously,
\[
\theta'_i(\varphi) \geq c_1 p^2 \theta'_{k-1}(\varphi) + \beta^k \geq \frac{1}{1 - c_1 p^2 \beta} \beta^k.
\]
Hence, $\theta_k = \theta_k(\varphi)$ is strictly monotone increasing function of $\varphi \in [0, \pi]$ satisfying $\theta'_k(\varphi)/\beta^k \in [d_1, d_2]$ with $d_1 = d_1(p, \beta, \varphi) = c_1 p^2 / \beta > 0$ and $d_2 = d_2(p, \beta, \varphi) = c_2 / (p^2 \beta) < \infty$.

For each $k \in \mathbb{N}$ and interval $I = (a, b) \subset [0, \pi]$, let
\[\varphi_k, 0 \leq a < \varphi_{k,1} < \varphi_{k,2} < \cdots < \varphi_{k,n_k} \leq b < \varphi_{k,n_k+1},\]
and let $n_k = n_k(I)$, be such that
\[
\theta_k(\varphi_{k,i}) = \left(\left\lfloor \frac{\theta_k(a)}{\pi} \right\rfloor + i\right) \pi
\]
holds for $i = 0, 1, \ldots, n_k + 1$ with $[x]$ the integer part of $x \in \mathbb{R}$. We set $I_{k,0} = (a, \varphi_{k,1})$, $I_{k,i} = (\varphi_{k,i}, \varphi_{k,i+1})$ for $i = 1, \ldots, n_k - 1$, $I_{k,n_k} = (\varphi_{k,i}, b]$ and note that $\{I_{k,i}\}_{i=0}^{n_k}$ is a partition of $I$ with
\[
|I_{k,i}| \in \left[\frac{\pi}{d_2 \beta^k}, \frac{\pi}{d_1 \beta^k}\right]
\]
for every $i \neq 0, n_k$. Let $\{\xi_k(\varphi)\}_{k \geq 1}$ be a sequence defined by the linear interpolation of $\theta_k$ at the points $\{\varphi_{k,i}\}$:
\[\xi_k(\varphi) = \theta_k(\varphi_{k,i}) + \frac{\varphi - \varphi_{k,i}}{|I_{k,i}|} \pi, \quad \varphi \in I_{k,i}.
\]

**Theorem 5.3.** $\{\xi_k(\varphi)\}_{k \geq 1}$ is u.d. mod $\pi$ for almost every $\varphi \in [0, \pi]$, any $p \in (0, 1)$ and $\beta \geq \beta_0(p, \varphi) > 2$ large enough.

**Proof.** To prove theorem 5.3, it is enough to verify the assumption of theorem 5.1. For $h \in \mathbb{Z}$ and $I = (a, b) \subset [0, \pi]$, we have
\[
\|S_h\|_{L^2([a, b])}^2 = \frac{1}{n^2} \sum_{k,m=1}^{n} \int_a^b \exp \{2ih (\xi_m(\varphi) - \xi_k(\varphi))\} \, dx \leq \frac{|b - a|}{n} + \frac{2}{n^2} \sum_{1 \leq k < m \leq n} \left| \int_a^b \exp \{2ih (\xi_m(\varphi) - \xi_k(\varphi))\} \, d\varphi \right|.
\]
Note that, by (5.12), $|I_{m,j}| < |I_{k,i}|$ holds uniformly for $k < m$ if $\beta$ is large enough and there is at least one $l = l(j)$ such that $I_{k,i} \cap I_{m,j} \neq \emptyset$. Using
\[
\int_{I_{m,j}} \exp \{2ih \xi_m(\varphi)\} \, d\varphi = \int_{I_{m,j}} \exp \left\{2ih \frac{\varphi - \varphi_{m,i}}{|I_{m,j}|} \pi \right\} \, d\varphi = |I_{m,j}| \int_0^1 e^{2ihx\pi} \, dx = 0
\]
together with \(|e^y - 1| \leq |y|\), for any \(y \in \mathbb{R}\) and (5.12),

\[
\left| \int_{I_{m,j}} \exp \left\{ 2i\hbar (\xi_m(\varphi) - \xi_k(\varphi)) \right\} d\varphi \right| = \left| \int_{I_{m,j}} \left( \exp \left\{ -2i\hbar \left( \frac{\xi_k(\varphi)}{|I_{k,j}|} \pi \right) \right\} - 1 \right) \times \exp \left\{ 2i\hbar \left( \xi_m(\varphi) + \frac{\varphi_{k,j}}{|I_{k,j}|} \pi \right) \right\} d\varphi \right| \\
\leq \frac{2|\hbar|\pi}{|I_{k,j}|} \int_{I_{m,j}} \varphi d\varphi = |h| \pi \frac{|I_{m,j}|^2}{|I_{k,j}|}
\]

and so

\[
\left| \int_{a}^{b} \exp \left\{ 2i\hbar (\xi_m(\varphi) - \xi_k(\varphi)) \right\} d\varphi \right| \leq \sum_{j=0}^{n_k} \left| \int_{I_{m,j}} \exp \left\{ 2i\hbar (\xi_m(\varphi) - \xi_k(\varphi)) \right\} d\varphi \right| \\
\leq |h| \pi \sup_{i} \frac{|I_{m,i}|}{|I_{k,j}|} \sum_{j=0}^{n_k} |I_{m,j}| \\
\leq \frac{|h| \pi d_2}{d_1} \beta^{m-k} |b-a|.
\]

Substituting (5.16) into (5.14) yields

\[
\|S_n\|_{L_2[a,b]}^2 \leq \left( 1 + \frac{2|\hbar| \pi d_2}{d_1 (\beta - 1)} \right) \frac{|b-a|}{n},
\]

which proves the assumption of theorem 5.1 and concludes the proof of the theorem. □

**Remark 5.4.** \(\left| I_{m,j} \right| < \left| I_{k,j} \right|\) holds uniformly for \(k < m\) if \(p^4 \beta > c_1/c_1\) (\(\approx 1\) for \(p\) small, see proposition 5.2). The condition has been used in the proof for the integral (5.15) over \(I_{m,j}\) to be uniformly bounded, including the intervals with \(j = 0, n_m\) at the extremities, which could be handled separately. Theorem 5.3 holds without this condition.

To estimate the (pointwise) distance between \(\xi_k(\varphi)\) and \(\theta_k(\varphi)\), let \(\left\{ \psi_{k,j} \right\}_{j=0}^{n_k}\) be the set of all values satisfying \(\xi_k(\varphi_{k,j}) = \theta_k(\varphi_{k,j})\). Note that this includes points that were not in the previous list, since \(\theta_k(\varphi_{k,j})\) may not necessarily be a multiple of \(\pi\). From (4.7), (5.13) and \(\left| I_{k,j} \right| = \psi_{k,j+1} - \psi_{k,j}\), this difference can be written as

\[
\theta_k(\varphi) - \xi_k(\varphi) = \beta_k \varphi + g(\theta_{k-1}(\varphi), \varphi) - \left( \frac{\varphi - \psi_{k,j}}{|I_{k,j}|} \theta_k(\psi_{k,j+1}) + \frac{\varphi_{k,j+1} - \varphi}{|I_{k,j}|} \theta_k(\psi_{k,j}) \right) \\
= g_{k-1}(\varphi) - \frac{\varphi - \psi_{k,j}}{|I_{k,j}|} g_{k-1}(\psi_{k,j+1}) - \frac{\varphi_{k,j+1} - \varphi}{|I_{k,j}|} g_{k-1}(\psi_{k,j})
\]

for \(\varphi \in I_{k,j}\). Here we make explicit the dependence of \(g\) on \(\varphi\) and use the shorthand \(g_n(\varphi) \equiv g(\theta_n(\varphi), \varphi)\).

**Proposition 5.5.**

\[
|\theta_k(\varphi) - \xi_k(\varphi)| \leq \frac{|I_{k,j}|}{4} \left( \max_{\varphi \in I_{k,j}} g'_{k-1}(\varphi) - \min_{\varphi \in I_{k,j}} g'_{k-1}(\varphi) \right)
\]

holds for \(\psi_{k,j} < \varphi \leq \psi_{k,j+1}\) and \(j = 0, \ldots, n_k\).
This is a refined (local) version of an upper bound given in appendix A of [Z]. We prove it in the case where \( g''_{k-1}(\varphi) \) does not vanish in \( I_{k,j} \). The general case follows in a similar way.

**Proof.** To maximize the difference (5.17) in absolute value, \( g_{k-1}(\varphi) \) is replaced by a piecewise linear continuous function:

\[
\begin{align*}
    h_{k-1}(\varphi) := &
    \begin{cases}
        h^{+}_{k-1}(\varphi) = g_{k-1}(\varphi_{k,j}) + g'_{k-1}(\varphi_{k,j}) \left( \varphi - \varphi_{k,j} \right) & \text{if } \varphi \in (\varphi_{k,j}, \bar{\varphi}), \\
        h^{-}_{k-1}(\varphi) = g_{k-1}(\varphi_{k,j+1}) - g'_{k-1}(\varphi_{k,j+1}) \left( \varphi - \varphi_{k,j+1} \right) & \text{if } \varphi \in (\bar{\varphi}, \varphi_{k,j+1}],
    \end{cases}
\end{align*}
\]

with \( \bar{\varphi} \) defined by continuity:

\[
    h_{k-1}(\bar{\varphi}) = h^{+}_{k-1}(\bar{\varphi}),
\]

which, after some manipulations, gives

\[
    \left| \theta_k(\varphi) - \bar{\xi}_k(\varphi) \right| \leq \bar{\varepsilon} \left( 1 - \bar{\varepsilon} \right) \left| I_{k,j} \right| \left| g'_{k-1}(\varphi_{k,j+1}) - g'_{k-1}(\varphi_{k,j}) \right|
\]

with \( \bar{\varepsilon} = (\bar{\varphi} - \varphi_k) / \left| I_{k,j} \right| \) and, together with \( \bar{\varepsilon} \left( 1 - \bar{\varepsilon} \right) \leq 1/4 \), proves the proposition. \( \square \)

**Theorem 5.6.** Let \( J \in \mathcal{A} \setminus \mathcal{A}_0 \) as stated in theorem 4.4. Conclusions (a) and (b) of theorem 4.4, with intervals \( I_1 \) and \( I_2 \) replaced by

\[
    I^*_1 \equiv \left\{ \lambda \in [-2, 2] : \left( \frac{4}{1 - p^2} \right)^2 (\beta - \epsilon) \left( 4 - \lambda^2 \right) > \epsilon \right\}
\]

and

\[
    I^*_2 \equiv \left\{ \lambda \in [-2, 2] : \left( \frac{4}{1 - p^2} \right)^2 (\beta^3 - \epsilon) \left( 4 - \lambda^2 \right) < \epsilon \right\},
\]

hold if the limit superior of (5.1) is bounded

\[
    E := \limsup_{N \to \infty} E_N \leq \epsilon
\]

for some \( \epsilon > 0 \). In addition, for each \( \epsilon > 0 \) and \( (p, \varphi) \in (0, 1) \times (0, \pi) \) there exist \( \beta_0 = \beta_0(\epsilon, p, \varphi) \) sufficiently large such that (5.20) holds for \( \beta \geq \beta_0 \).

**Proof.** By (3.10), (5.12), (5.13) and proposition 5.5

\[
    \frac{1}{N} \sum_{k=1}^{N_0} \left| f(\theta_k) - f(\xi_k) \right| \leq \max_{\theta \in [0, \pi]} \left| f'(\theta) \right| \frac{1}{N} \sum_{k=1}^{N_0} |\theta_k - \xi_k| 
\]

converges to 0 as \( N \to \infty \) for any \( N_0 \) and it does not contribute to \( E \). As a consequence, the contribution of partial derivative \( \partial g / \partial \varphi \) to \( g''_k(\varphi) \) may, without loss, be disregarded from the estimate (5.18) for \( k \geq N_0 \) and the error absorbed into the constant \( C_N \) of (4.19). This, together with the fact that \( \theta_k(\varphi) \simeq g(\beta^{k-1} \varphi, \varphi) + \beta^k \varphi \) for \( \beta \) and \( k \) large enough, leads to the following scaling version of (5.18).

Let \( \left\{ \theta^i \right\}_{i=0}^{n+1} \) be a sequence of numbers satisfying \( \theta_0 = -\pi/2 < \theta_1 \leq \cdots < \theta_n \leq \pi/2 < \theta_{n+1} \) and such that the intervals \( I^0 = [-\pi/2, \theta^1], I^1 = (\theta^1, \theta^{i+1}], I^n = (\theta^n, \pi/2] \) form a partition of \([-\pi/2, \pi/2] \) satisfying

\[
    1 - \delta_1 \leq \beta \left| I^i \right| \leq 1 + \delta_2
\]

for some small fixed constants \( \delta_1 \) and \( \delta_2 \). If \( y : [-\pi/2, \pi/2] \times [0, \pi] \to \mathbb{R} \) is the continuous piecewise linear function defined by

\[
    y(\theta^i, \varphi) = g(\theta^i, \varphi), \quad i = 0, \ldots, n + 1,
\]
we have, analogously to the estimate \((5.18)\),
\[
|y(\theta, \varphi) - g(\theta, \varphi)| \leq \left| \frac{f'}{4} \right| \max_{\theta \in \Gamma} \frac{\partial g}{\partial \theta}(\theta, \varphi) - \min_{\theta \in \Gamma} \frac{\partial g}{\partial \theta}(\theta, \varphi)
\]
(5.22)
for \(\theta \in \Gamma\) which, in view of \((5.21)\) and \((5.5)\), is bounded from above by \(C/\beta\) for any \((p, \varphi) \in (0, 1) \times (0, \pi)\) provided \(\beta\) is sufficiently large.

By the fundamental theorem of calculus together with the scaling version \((5.22)\) of \((5.18)\), for every \(k > N_0\)
\[
|f(\theta_k) - f(\xi_k)| \leq \sup_{\min(\theta_k, \xi_k) < \theta < \max(\theta_k, \xi_k)} |f'(\theta)| |\theta_k - \xi_k|
\]
\[
\leq \sup_{0 < \theta < \pi} \left| f'(\theta) \right| \max_{\theta \in \Gamma} \left| y(\theta, \varphi) - g(\theta, \varphi) \right| < \varepsilon
\]
holds for every \(\varepsilon > 0\) provided \(\beta > \beta_0\) with \(\beta_0 = \beta_0(p, \varphi, \varepsilon)\) sufficiently large. The spectral conditions defining \(I_1^\varepsilon\) and \(I_2^\varepsilon\) are obtained as in the proof of theorem \(4.4\) with the inequalities \(\beta/r > 1\) and \(\beta r (\beta/r)^2 < 1\) substituted by \(\beta e^{-\varepsilon}/r > 1\) and \(\beta e^{-\varepsilon}/r^2 < 1\), respectively. This concludes the proof of theorem \(5.6\).

**Remark 5.7.** For each \(\varepsilon > 0\) and \((p, \varphi) \in (0, 1) \times (0, \pi)\) there exists \(\beta_1 = \beta_1(\varepsilon, p, \varphi)\) sufficiently large such that \(I_1^\varepsilon\) given by \((5.19)\) is not an empty set for any \(\beta \geq \beta_1\).

We are now able to state the main result of this section. Theorem \(5.6\) together with remark \(5.7\) implies the following remark.

**Theorem 5.8.** Let \(J \in \mathcal{H}\) as stated in theorem \(4.4\) and let \(\bar{\beta} = \bar{\beta}(\varepsilon, p, \varphi) = \max(\beta_0, \beta_1)\) for each \(\varepsilon > 0\) and \((p, \varphi) \in (0, 1) \times (0, \pi)\), with \(\beta_0, \beta_1\) as in theorem \(5.6\) and remark \(5.7\). Then, for every \(\beta \geq \bar{\beta}\) there exists a set \(\Lambda_1^\varepsilon\) of Lebesgue measure zero such that the spectrum \(\sigma(J)\) restricted to set \(I_1^\varepsilon \setminus \Lambda_1^\varepsilon\) is purely s.c.

Concerning \(I_2^\varepsilon\), we have, however, the following remark.

**Remark 5.9.**
1. \(f\) given by \((3.10)\) has first derivative extremized by
\[
f'(\theta^-) \leq f'(\theta) \leq f'(\theta^+) = -f'(\theta^-) = \frac{2 |b| |c|}{\sqrt{a^2 + b^2}},
\]
(5.24)
where \(a, b\) and \(c\) are given in \((4.14)\). When \(\varphi\) approaches the boundary of \((0, \pi)\) the upper bound \((5.24)\) tends to infinity, and this turns \(I_2^\varepsilon\) into the empty set! More precisely, \((5.23)\) and \((5.24)\) imply that
\[
\left( \frac{p}{1 - p^2} \right)^2 \frac{\beta}{\cot^2 \varphi} \geq \frac{C}{\varepsilon}
\]
holds for some \(C > 0\) if \((4 - \lambda^2)/\lambda^2 = \cot^2 \varphi\) is sufficiently small and this goes in the opposite direction of the inequality in \(I_2^\varepsilon\).
2. For \(\beta\) a fixed large integer and \(p\) very small, the scaling description of \((5.18)\) requires partitions which do not satisfy \((5.5)\). According to proposition \(5.2\), there is a small interval \(J\) containing the unstable fixed point \(\theta^u = \varphi - \pi/2\) such that \(g\) changes abruptly from \(-\pi/2 + O(p^2)\) to \(\pi/2 - O(p^2)\) and every partition interval there has to be of order \(p^2\). Estimating \((5.22)\) by \(C p^2\), together with \((5.23)\) and \((5.24)\), does not yield \((5.20)\) for any \(\varepsilon > 0\) by choosing \(p\) small enough without evoking statistical properties of the sequence \(\{\theta_k\}\). In other words, the sign changes of \((5.1)\) have to be taken into account.

Thus, by remark \(5.9\), the p.p. spectrum does not seem to be tractable by perturbative methods. We turn to this problem in the next section.
6. Numerical evidence for asymptotic distribution function

In this section we present the compelling numerical evidence that the Pr"ufer angles have an a.d.f. mod $\pi$ for almost every $\varphi \in [0, \pi]$. We have computed the asymptotic density function $\rho = \text{dv}/\text{d}\varphi$ for $\beta = 2, \varphi$ a typically dyadic irrational and $p^2$ varying over the values 0.01, 0.1 and 0.3. The corresponding histograms presented in figure 4, each containing $10^7$ iteration points of the map (4.7), indicate that the sequence $\left(\theta_k\right)_{k \geq 1}$ has continuous a.d.f. mod $\pi$, $\nu(\theta)$ which tends to the identity $\theta$ when $p \to 0$. These claims will now be examined numerically using two criteria whose background is given in the following.

The powerful metric result is ( [KN], theorem 4.3 of chapter 1) the following theorem:

**Theorem 6.1 (Koksma’s metric theorem).** Let $\left(\xi_k(x)\right)_{k \geq 1}$ be a sequence of real valued continuously differentiable functions defined in an interval $[a, b]$. If for any two positive integers $k \neq m$, $\xi'_k - \xi'_m$ is a monotone function of $x$ such that $|\xi'_k - \xi'_m| \geq K > 0$ holds uniformly in $k, m$ and $x \in [a, b]$, then $\left(\xi_k\right)_{k \geq 1}$ is u.d. mod $\pi$ for almost all $x \in [a, b]$.

Since $\xi'_k(\varphi) = 0$ for almost every $\varphi \in [0, \pi]$, theorem 6.1 applies to $\left(\xi_k\right)_{k \geq 1}$ given by (5.13) but cannot be applied for the sequence $(\theta_k)_{k \geq 1}$ with $p \in (0, 1)$. To see this and make our numerical results understandable the proof of Koksma’s theorem is included.

**Proof.** By the second mean value theorem, there exists $c$ such that $a < c < b$ and

$$\int_a^b e^{2i(\xi_k(x) - \xi_m(x))} \text{d}x = \left| \frac{1}{2i} \int_a^b \frac{de^{2i(\xi_k(x) - \xi_m(x))}}{\xi'_k(x) - \xi'_m(x)} \right|$$

$$= \left| \frac{1}{2i} \left( \frac{1}{\xi'_k(c) - \xi'_m(c)} \int_a^c \text{d}e^{2i(\xi_k(x) - \xi_m(x))} + \frac{1}{\xi'_k(b) - \xi'_m(b)} \int_c^b \text{d}e^{2i(\xi_k(x) - \xi_m(x))} \right) \right|$$

$$\leq \left| \frac{1}{|\xi'_k(c) - \xi'_m(c)|} + \frac{1}{|\xi'_k(b) - \xi'_m(b)|} \right|.$$

Ordering $\left\{\xi'_{k_i}(x)\right\}_{i=1}^n$ such that $\xi'_{k_1}(x) < \xi'_{k_2}(x) < \cdots < \xi'_{k_n}(x)$ holds for all $x$, the difference in any two consecutive numbers satisfies $\xi'_{k_i}(x) - \xi'_{k_{i-1}}(x) > K$ and we have

$$\sum_{m=1}^{n-1} \frac{1}{\xi'_{k_i}(x) - \xi'_{k_m}(x)} < 2 \sum_{m=1}^{n} \frac{1}{mK} < \frac{2}{K} \ln 3n.$$

This together with (5.14) yields

$$\|S_n\|_{L^2([a,b])} < \frac{|b - a|}{n} + \frac{4}{nK} \ln 3n,$$

which combined with theorem 5.1 concludes the proof. $\square$

In order to apply theorem 6.1, $\theta'_k(\varphi) - \theta'_m(\varphi)$ has to be a monotone function for each interval $[a, b]$ independent of $k$ and $m$. But this is not true for the sequence (4.7) since the number of monotone intervals of $\theta'_k - \theta'_m$ in $I \subset (0, \pi)$, with $k < m$ fixed, for which we may apply Koksma’s theorem, is proportional to $p^2$ and this number is compensated by the constant $K$ of theorem 6.1 which, in view of equations (5.9) and (5.3), has the same $k$ dependence:

$$\theta'_k(\varphi) - \theta'_{k-1}(\varphi) = (g'(\theta_{k-1}) - 1) \theta'_{k-1}(\varphi) + \beta^k$$

$$\geq (1 - p^2) \theta'_{k-1} + \beta^k$$

$$\geq \frac{\beta p^2 - c_2 - p^2 + p^4}{p^2 \beta - c_2} \beta^k.$$
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Figure 5. Plots of $\|S_n\|_{L^2(I)}$ for $\beta = 2$ and $p = 0.3$

According to the results shown in figure 4, $S_n(\varphi)$ given by (5.2) with $\xi_k$ replaced by $\theta_k(\varphi)$ cannot converge to 0 for any $\varphi \in [0, \pi]$ if $p \in (0, 1)$. Note that, by the Weyl criterion together with (4.8),

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} e^{2i\theta_k(\varphi)} = \int e^{2i\theta} d\nu(\theta) \equiv \omega_h \quad (6.3)$$

converges to 0 for every $h \neq 0$ if and only if $(\theta_k)_{k \geq 1}$ is uniformly distributed, i.e. $\nu(\theta) = \theta$.

Despite this, we have numerically tested the criterion of theorem 5.1 for the sequence $(\theta_k)_{k \geq 1}$. Figure 5 plots $\|S_n\|_{L^2(I)}$ for $\beta = 2$ and $p = 0.3$ as $n$ varies from $10^3$ to $10^4$. We have used the Monte Carlo method of integration for an interval $I$ of length $2^{-990}$ around $\varphi = 0.89375$ with the standard deviation from the predicted value kept below 2%. According to the best interpolation fit $\|S_n\|_{L^2(I)}$ approaches a constant as fast as $c_1 - c_2 n^{-0.9}$. For sequences satisfying the assumptions of theorem 6.1 (see equation (6.2)), $\|S_n\|_{L^2(I)}^2$ converges to 0 as fast as $cn^{-1} \ln n$. For the additive part of the sequence (4.7), $x_k(\varphi) = \beta^k \varphi$, or for the sequence $\xi_k(\varphi)$, the situation is even better with $\|S_n\|_{L^2(I)}^2$ decaying as $1/n$ without the log correction.

An extension of Koksma’s metric theorem to a nonuniform distribution $\nu$ is an open problem. However, if somehow one is able to subtract its limit point $\omega_h$ from $S_n$, the integral on the second line of (6.2) would, instead of behaving like (6.2), decay as fast as $c_2 n^{-0.9}$ according to the behaviour of figure 5 for $c_1 - \|S_n\|_{L^2(I)}^2$. Hence, the decay may be interpreted as an indication that the double sum in (5.14) would grow like $n^{1+\delta}$, $\delta \simeq 0.1$, and the sequence $(\theta_k)_{k \geq 1}$ would have continuous a.d.f. mod $\pi$ for almost every $\varphi \in [0, \pi]$.

Finally, we verify the following criterion due to Wiener–Schoenberg (see theorem 7.5, chapter 1 of [KN]): $(\theta_k)_{k \geq 1}$ has a continuous a.d.f. mod $\pi$ if and only if $\omega_h = \lim_{n \to \infty} S_n$ exists for every positive integer $h$ and

$$\lim_{H \to \infty} \frac{1}{H} \sum_{h=1}^{H} |\omega_h|^2 = 0.$$

Figure 6 is a log–log plot of $\sum_{h=1}^{H} |\omega_h|^2 / H$, with each point $\omega_h$ obtained by $10^7$ iteration of (4.7) and $\varphi$ a fixed dyadic irrational, as $H$ varies from 2 to 6000.
7. Conclusion: open problems

In this paper we have introduced a new method to study the spectrum of sparse Jacobi matrices, based on u.d. (of the Prüfer angles \((\theta_k)_{k \geq 1}\)) on the space-variable rather than on the energy. Together with metric theorems, we were able to prove the existence of a s.c. spectrum (theorem 5.8). The proof of a (dense) p.p. spectrum seems, however, not to be accessible to a perturbation treatment (remark 5.9), but under an assumption (conjecture 4.1) on the distribution of \((\theta_k)_{k \geq 1}\), theorem 4.4 yields the existence of dense p.p. for a complementary set of parameters and thus of a spectral transition.

In section 6 strong numerical evidence for conjecture 4.1—again supported by the metric theorems—was provided. Its verification remains, however, an open problem in the theory of nonlinear dynamical systems (n.d.s.). In fact, the Prüfer angles are associated with a n.d.s. (3.11), (3.12) and (3.13) in the space-variable. This n.d.s. belongs to the class of perturbations of the \(\beta\)-adic shift:

\[
x_k = \beta x_{k-1} \mod 1, \quad \beta \geq 2 \text{ integer},
\]

when the \(a_k\) in (3.11) satisfy (4.5) (see also 4.7). Such perturbations have been considered in order to investigate whether arithmetical numbers are normal to the base \(\beta\) (see [BC, L] and references therein) and are thus of independent interest. We hope that the present paper may stimulate further progress in the ergodic properties of these n.d.s.

Acknowledgments

We thank Professors B Simon, A Klein, A Lopes and the anonymous referee for instructive comments. WW should like to thank Professors J S Howland and A Joye for fruitful conversations and remarks. DHUM is partially supported by CNpq and FAPESP, WW by CNPq and LFG by FAPESP under Grant 03/13772-0.

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