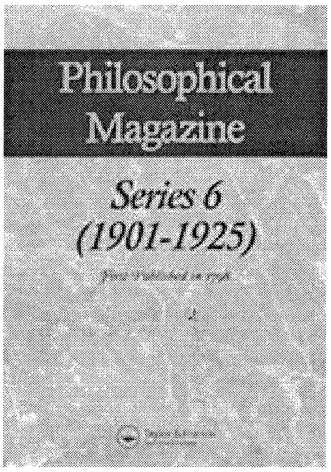

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VIII. *On the pressure developed in a liquid during the collapse of a spherical cavity*

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much. So the deflexion for a ray of starlight grazing the sun is

$$2\times\left(\frac{250}{185000}\right)^2=3\cdot6\times10^{-6},\text{ or }0''\cdot74.$$

Longitudinal Possibility.

The velocity of light issuing radially from a body might on this hypothesis also be affected, since it could be pulled back by a maximum amount represented by the free fall from infinity, *i. e.* by $\sqrt{(2qgR)}$; though the longitudinal q need not be the same as the transverse q concerned in deflexion.

But this sort of action, if it can be imagined as likely to occur, and even if it caused a reduction in sunlight-velocity of 26 miles a second in the neighbourhood of the earth, would not yield any Doppler effect; the waves would still be received at their emitted frequency.

VIII. *On the Pressure developed in a Liquid during the Collapse of a Spherical Cavity.* By Lord RAYLEIGH, *O.M., F.R.S.**

WHEN reading O. Reynolds's description of the sounds emitted by water in a kettle as it comes to the boil, and their explanation as due to the partial or complete collapse of bubbles as they rise through cooler water, I proposed to myself a further consideration of the problem thus presented; but I had not gone far when I learned from Sir C. Parsons that he also was interested in the same question in connexion with cavitation behind screw-propellers, and that at his instigation Mr. S. Cook, on the basis of an investigation by Besant, had calculated the pressure developed when the collapse is suddenly arrested by impact against a rigid concentric obstacle. During the collapse the fluid is regarded as incompressible.

In the present note I have given a simpler derivation of Besant's results, and have extended the calculation to find the pressure in the interior of the fluid during the collapse. It appears that before the cavity is closed these pressures may rise very high in the fluid near the inner boundary.

* Communicated by the Author.

As formulated by Besant *, the problem is—

"An infinite mass of homogeneous incompressible fluid acted upon by no forces is at rest, and a spherical portion of the fluid is suddenly annihilated; it is required to find the instantaneous alteration of pressure at any point of the mass, and the time in which the cavity will be filled up, the pressure at an infinite distance being supposed to remain constant."

Since the fluid is incompressible, the whole motion is determined by that of the inner boundary. If U be the velocity and R the radius of the boundary at time t, and u the simultaneous velocity at any distance r (greater than R) from the centre, then

$$u/U = R^2/r^2 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and if ρ be the density, the whole kinetic energy of the motion is

$$\tfrac{1}{2}\rho \int_R^\infty u^2 \, . \, 4\pi r^2 dr = 2\pi\rho U^2 R^3. \quad . \quad . \quad . \quad (2)$$

Again, if P be the pressure at infinity and R_0 the initial value of R, the work done is

$$\frac{4\pi P}{3} (R_0^3 - R^3). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

When we equate (2) and (3) we get

$$U^2 = \frac{2P}{3\rho} \left(\frac{R_0^3}{R^3} - 1 \right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

expressing the velocity of the boundary in terms of the radius. Also, since $U = dR/dt$,

$$t = \sqrt{\left(\frac{3\rho}{2P}\right)} \cdot \int_R^{R_0} \frac{(R^{3/2} dR}{(R_0^3 - R^3)^{\frac{1}{2}}} = R_0 \sqrt{\left(\frac{3\rho}{2P}\right)} \cdot \int_\beta^1 \frac{\beta^{3/2} d\beta}{(1-\beta^3)^{\frac{1}{2}}} \quad . \quad . \quad . \quad (5)$$

if $\beta = R/R_0$. The time of collapse to a given fraction of the original radius is thus proportional to $R_0 \rho^{\frac{1}{2}} P^{-\frac{1}{2}}$, a result which might have been anticipated by a consideration of "dimensions." The time τ of *complete* collapse is obtained by making $\beta = 0$ in (5). An equivalent expression is given by Besant, who refers to Cambridge Senate House Problems of 1847.

* Besant's 'Hydrostatics and Hydrodynamics,' 1859, § 158.

Writing $\beta^3=z$, we have

$$\int_0^1 \frac{\beta^{3/2}d\beta}{(1-\beta^3)^{\frac{1}{2}}} = \frac{1}{3}\int_0^1 z^{-\frac{1}{6}}(1-z)^{-\frac{1}{2}}\,dz,$$

which may be expressed by means of Γ functions. Thus

$$\tau = R_0\sqrt{\left(\frac{\rho}{6P}\right)}\cdot\frac{\Gamma(\frac{5}{6})\,.\,\Gamma(\frac{1}{2})}{\Gamma(\frac{4}{3})} = \cdot 91468\,R_0\sqrt{(\rho/P)}. \quad . \quad (6)$$

According to (4) U increases without limit as R diminishes. This indefinite increase may be obviated if we introduce, instead of an internal pressure zero or constant, one which increases with sufficient rapidity. We may suppose such a pressure due to a permanent gas obedient to Boyle's law. Then, if the initial pressure be Q, the work of compression is $4\pi Q R_0^3 \log(R_0/R)$, which is to be subtracted from (3). Hence

$$U^2 = \frac{2P}{3\rho}\left(\frac{R_0^3}{R^3}-1\right) - \frac{2Q}{\rho}\frac{R_0^3}{R^3}\log\frac{R_0}{R}; \quad . \quad . \quad . \quad (7)$$

and U=0 when

$$P(1-z)+Q\log z=0, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

z denoting (as before) the ratio of volumes R^3/R_0^3. Whatever be the (positive) value of Q, U comes again to zero before complete collapse, and if $Q>P$ the first movement of the boundary is outwards. The boundary oscillates between two positions, of which one is the initial.

The following values of P/Q are calculated from (8):

z.	P/Q.	*z*.	P/Q.
$\frac{1}{1000}$	6·9147	1	arbitrary
$\frac{1}{100}$	4·6517	2	0·6931
$\frac{1}{10}$	2·5584	4	0·4621
$\frac{1}{4}$	1·8484	10	0·2558
$\frac{1}{2}$	1·3863	100	0·0465
1	arbitrary	1000	0·0069

Reverting to the case where the pressure inside the cavity is zero, or at any rate constant, we may proceed to calculate the pressure at any internal point. The general equation of pressure is

$$\frac{1}{\rho}\frac{dp}{dr}=-\frac{Du}{Dt}=-\frac{du}{dt}-u\frac{du}{dr}, \quad . \quad . \quad . \quad (9)$$

u being a function of r and t, reckoned positive in the direction of increasing r. As in (1), $u=UR^2/r^2$, and

$$\frac{du}{dt}=\frac{1}{r^2}\frac{d}{dt}(UR^2).$$

Also

$$\frac{d(U^2R)}{dt}=2R\frac{dR}{dt}U+R^2\frac{dU}{dt}=2RU^2+R^2\frac{dU}{dt},$$

and by (4)

$$\frac{dU}{dt}=-\frac{P}{\rho}\frac{R_0^3}{R^4},$$

so that

$$\frac{d(U^2R)}{dt}=2RU^2-\frac{P}{\rho}\frac{R_0^3}{R^2}.$$

Thus, suitably determining the constant of integration, we get

$$\frac{p}{P}-1=\frac{R}{3r}\left\{\frac{R_0^3}{R^3}-4\right\}-\frac{R^4}{3r^4}\left\{\frac{R_0^3}{R^3}-1\right\}. \quad . \quad (10)$$

At the first moment after release, when $R=R_0$, we have

$$p=P(1-R/r). \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

When $r=R$, that is on the boundary, $p=0$, whatever R may be, in accordance with assumptions already made.

Initially the maximum p is at infinity, but as the contraction proceeds, this ceases to be true. If we introduce z as before to represent R_0^3/R^3, (10) may be written

$$\frac{p}{P}-1=\frac{R}{3r}(z-4)-\frac{R^4}{3r^4}(z-1), \quad . \quad . \quad . \quad (12)$$

and

$$\frac{dp/P}{dr}=\frac{R}{3r^2}\left\{\frac{(4z-4)R^3}{r^3}-(z-4)\right\}. \quad . \quad (13)$$

The maximum value of p occurs when

$$\frac{r^3}{R^3}=\frac{4z-4}{z-4}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and then

$$\frac{p}{P}=1+\frac{(z-4)R}{4r}=1+\frac{(z-4)^{\frac{4}{3}}}{4^{\frac{4}{3}}(z-1)^{\frac{1}{3}}} \quad . \quad . \quad (15)$$

So long as z, which always exceeds 1, is less than 4, the greatest value of p, viz. P, occurs at infinity; but when z exceeds 4, the maximum p occurs at a finite distance given by (14) and is greater than P. As the cavity fills up, z becomes great, and (15) approximates to

$$\frac{p}{P}=\frac{z}{4^{\frac{4}{3}}}=\frac{R_0^3}{4^{\frac{4}{3}}R^3}, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

corresponding to

$$r=4^{\frac{1}{3}}R=1\cdot587R. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

It appears from (16) that before complete collapse the pressure near the boundary becomes very great. For example, if $R=\frac{1}{20}R_0$, $p=1260P$.

This pressure occurs at a relatively moderate distance outside the boundary. At the boundary itself the pressure is zero, so long as the motion is free. Mr. Cook considers the pressure here developed when the fluid strikes an absolutely rigid sphere of radius R. If the supposition of incompressibility is still maintained, an infinite pressure momentarily results; but if at this stage we admit compressibility, the instantaneous pressure P′ is finite, and is given by the equation

$$\frac{P'^2}{2\beta'}=\tfrac{1}{2}\rho U^2=\frac{P}{3}\left(\frac{R_0^3}{R^3}-1\right), \quad . \quad . \quad . \quad (18)$$

β' being the coefficient of compressibility. P, P′, β' may all be expressed in atmospheres. Taking (as for water) $\beta'=20{,}000$, $P=1$, and $R=\frac{1}{20}R_0$, Cook finds

$$P'=10300 \text{ atmospheres}=68 \text{ tons per sq. inch},$$

and it would seem that this conclusion is not greatly affected by the neglect of compressibility before impact.

The subsequent course of events might be traced as in 'Theory of Sound,' § 279, but it would seem that for a satisfactory theory compressibility would have to be taken into account at an earlier stage.

April 13, 1917.