4 Classical aerofoil theory

4.1. Introduction

Let us begin by noting some of the key events in the early days of aerodynamics.

1894 F. W. Lanchester presents a paper, 'The soaring of birds and the possibilities of mechanical flight', to a meeting of the Birmingham Natural History and Philosophical Society. It contains the elements of the circulation theory of lift, but not in conventional terms.

1897 Lanchester submits a written version of his paper for publication by the Physical Society. It is rejected.

1901 The Wright brothers encounter failure with their first attempts at glider design. One of them is heard to mutter that 'nobody will fly for a thousand years'.

1902 Kutta publishes a short paper, 'Lifting forces in flowing fluids'. It contains the solution for 2-D irrotational flow past a circular arc, with circulation round the surface and a finite velocity at the trailing edge (Exercise 4.8). The connection between circulation and lift is recognized, though not in the form of the general theorem (1.35).

1903 17 December: The Wright brothers achieve their first powered flight. It lasts for 12 seconds, although they improve on this later the same day.

1904 Prandtl presents his paper on boundary layers to the Third International Congress of Mathematicians at Heidelberg (see §8.1).

1906 Joukowski publishes the lift theorem (1.35):

If an irrotational two-dimensional fluid current, having at infinity the velocity V_∞, surrounds any closed contour on which the circulation of velocity is Γ, the force of the aerodynamic pressure acts on this contour in a direction perpendicular to the velocity and has the value

$$L' = \rho_\infty V_\infty \Gamma.$$

The direction of this force is found by causing to rotate through a right angle the vector V_∞ around its origin, in an inverse direction to that of the circulation.

1907 Lanchester publishes his *Aerodynamics*, although some of the most important results in the book date from as early as 1892. He was certainly years ahead of everyone else in recognizing the inevitability, and the importance, of trailing vortices from the tip of a wing of finite length (§1.7).

A list like this is a concise way of presenting some of the facts, but it can be misleading, for the events within it were, at the time, almost wholly unconnected. Thus Lanchester, Kutta, and Joukowski came to their various conclusions about aerodynamics quite independently, and Wilbur Wright, had he known, would probably not have had much time for any of them. He and his brother relied greatly on their own experimental work on wind-tunnel flows past aerofoils of various shapes, but as late as 1909 he wrote to Lanchester:

> . . . I note such differences of information, theory, and even ideals, as to make it quite out of the question to reach common ground . . . , so I think it will save me much time if I follow my usual plan and let the truth make itself apparent in actual practice.

Our first aim in this chapter is to establish that for uniform irrotational flow past an aerofoil with a sharp trailing edge there is just one value of the circulation Γ for which the velocity is finite everywhere (Kutta–Joukowski condition). In particular, we seek to show that in the case of a thin symmetrical aerofoil of length L making an angle of attack α with the oncoming stream the value Γ is given by

$$\Gamma = -\pi UL \sin \alpha. \tag{4.1}$$

We set about doing this by first solving the comparatively easy problem of irrotational flow past a circular cylinder, and then using the method of conformal mapping to infer the irrotational flow past 2-D objects of more wing-like cross-section.

We must add one important warning before we start. The present chapter is full of irrotational flows which involve slip at rigid boundaries. While any particular flow may well serve a quite different purpose, *it will represent correctly the motion of a viscous fluid at high Reynolds number only if the slip velocity can* *be adjusted to zero successfully, by a viscous boundary layer, without separation.* Rough guidelines on whether or not separation will occur have already been presented in §2.1.

4.2. Velocity potential and stream function

The velocity potential

The velocity potential ϕ is something that exists *only if* $\nabla \wedge \boldsymbol{u} = 0$; it is defined at any point P by

$$\phi = \int_{O}^{P} \boldsymbol{u} \cdot d\boldsymbol{x} \tag{4.2}$$

where O is some arbitrary fixed point. In a simply connected fluid region ϕ is independent of the path between O and P, and thus a single-valued function of position (Exercise 4.1.) Partial differentiation of eqn (4.2) gives

$$\boldsymbol{u} = \nabla \phi, \tag{4.3}$$

and the vector identity (A.2) at once confirms that this flow is irrotational, as desired.

This representation of an irrotational flow, eqn (4.3), is valid also in multiply connected fluid regions, but the integral in eqn (4.2) may then depend on the path from O to P, in which case ϕ will be a multivalued function of position. In this case, it is worth noting at once that the circulation round any closed curve C in the flow is given by

$$\Gamma = \oint_C \boldsymbol{u} \cdot d\boldsymbol{x} = \oint_C \nabla \phi \cdot d\boldsymbol{x} = [\phi]_C, \tag{4.4}$$

where the last expression denotes the change (if any) in ϕ after one circuit round C (see eqn (A.12)).

Let us take some examples. The uniform flow $\boldsymbol{u} = (U, 0, 0)$ has velocity potential $\phi = Ux$ (plus an insignificant arbitrary constant, which has no effect on the flow (4.3)). The stagnation point flow of Exercise 1.7:

$$u = \alpha x, \qquad v = -\alpha y, \qquad w = 0$$

is irrotational, and writing

$$\partial \phi / \partial x = \alpha x, \qquad \partial \phi / \partial y = -\alpha y, \qquad \partial \phi / \partial z = 0$$

we may integrate to obtain

$$\phi = \tfrac{1}{2}\alpha(x^2 - y^2).$$

In both these cases ϕ is a single-valued function of position; there is therefore no circulation round any closed circuit lying in the flow domain.

Now take the line vortex flow (1.21):

$$\boldsymbol{u} = \frac{k}{r}\,\boldsymbol{e}_\theta,$$

which is an irrotational flow *except at the origin*, where it is not defined. To meet this difficulty, consider the flow domain to be $r \geqslant a$, which is not simply connected, for there are now some closed curves (i.e. those which enclose $r = a$) which cannot be shrunk to a point without leaving the flow domain. To find the velocity potential we integrate

$$\frac{\partial \phi}{\partial r} = 0, \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{k}{r}, \qquad \frac{\partial \phi}{\partial z} = 0,$$

and thus obtain

$$\phi = k\theta,$$

which is a multivalued function of position. As we go round any circuit not enclosing $r = a$ it is clear that θ, and hence ϕ, will return, at the end of that circuit, to its original value. There is therefore no circulation round such a circuit. But as we go round any closed curve which winds once round the cylinder $r = a$, θ increases by 2π, and the circulation round such a circuit will therefore be $\Gamma = 2\pi k$. Thus all circuits which wind once round the cylinder have the same circulation (cf. Exercise 1.6).

The stream function

This is a useful device for representing flows which are *incompressible and two-dimensional*. The essential idea is to write

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \tag{4.5}$$

thus automatically satisfying the 2-D incompressibility condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4.6}$$

That such a function $\psi(x, y, t)$ may be found can be shown by a similar argument to that used above (Exercise 4.1).

An important property of ψ follows immediately from eqn (4.5), for

$$(\boldsymbol{u} \cdot \nabla)\psi = u\frac{\partial \psi}{\partial x} + v\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} = 0, \quad (4.7)$$

so ψ *is constant along a streamline*. This gives an effective way of finding the streamlines for a 2-D incompressible flow; if we can just find $\psi(x, y, t)$ the equations for the streamlines can be written down immediately.

A useful way of viewing the representation (4.5) is as

$$\boldsymbol{u} = \nabla \wedge (\psi \boldsymbol{k}), \quad (4.8)$$

where $\boldsymbol{k}$ is the unit vector in the z-direction. It provides, in particular, a way of obtaining the plane polar counterparts to eqn (4.5). Regarding ψ instead as a function of r, θ, and t, we obtain at once

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}, \quad (4.9)$$

and such a flow automatically satisfies the 2-D incompressibility condition in plane polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 \quad (4.10)$$

(see eqn (A.35)).

4.3. The complex potential

Suppose now that we have a flow which is (i) two-dimensional, (ii) incompressible, and (iii) irrotational. Then the velocity field can be represented by both eqns (4.3) and (4.5), so that

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}. \quad (4.11)$$

The second of the equations in each pair constitute the well known Cauchy–Riemann equations of complex variable theory, and provided that the partial derivatives in eqn (4.11) are continuous it follows that

$$w = \phi + i\psi \tag{4.12}$$

is an analytic function of the complex variable $z = x + iy$ (Priestley 1985, pp. 16, 184). We call $w(z)$ the *complex potential.*

One of the most important properties of a 2-D incompressible, irrotational flow is that its velocity potential and stream function both satisfy Laplace's equation, so

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{4.13}$$

and

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0, \tag{4.14}$$

as may be seen directly from eqn (4.11).

The velocity components u and v are directly related to dw/dz, which is most conveniently calculated as follows:

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv. \tag{4.15}$$

(Note the negative sign.) The flow speed at any point is therefore

$$q = (u^2 + v^2)^{\frac{1}{2}} = \left| \frac{dw}{dz} \right|. \tag{4.16}$$

We now consider a number of examples.

Uniform flow at an angle α to the x-axis

Here

$$u = U\cos\alpha, \qquad v = U\sin\alpha,$$

so $dw/dz = Ue^{-i\alpha}$, and therefore

$$w = Uze^{-i\alpha}. \tag{4.17}$$

Line vortex

We may write this flow as

$$\boldsymbol{u} = \frac{\Gamma}{2\pi r}\boldsymbol{e}_\theta, \tag{4.18}$$

where Γ is the circulation round any simple circuit enclosing the vortex, and we already know from the previous section that

$$\phi = \Gamma\theta/2\pi. \tag{4.19}$$

Using eqn (4.9) we may also write

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0, \qquad -\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\pi r},$$

whence

$$\psi = -\frac{\Gamma}{2\pi}\log r.$$

Thus

$$\phi + \mathrm{i}\psi = \frac{\Gamma}{2\pi}(\theta - \mathrm{i}\log r) = -\frac{\mathrm{i}\Gamma}{2\pi}(\log r + \mathrm{i}\theta),$$

and the complex potential for a line vortex at the origin is therefore

$$w = -\frac{\mathrm{i}\Gamma}{2\pi}\log z. \tag{4.20}$$

By the same token, the complex potential for a line vortex at $z = z_0$ is

$$w = -\frac{\mathrm{i}\Gamma}{2\pi}\log(z - z_0). \tag{4.21}$$

2-D irrotational flow near a stagnation point

If the complex potential $w(z)$ is analytic in some region it will possess a Taylor series expansion in the neighbourhood of any point z_0 in that region (Priestley 1985, p. 69), i.e.

$$w(z) = w(z_0) + (z - z_0)w'(z_0) + \tfrac{1}{2}(z - z_0)^2 w''(z_0) + \dots .$$

Now, the first term is an inconsequential constant which makes no difference to $\mathrm{d}w/\mathrm{d}z$, and if $z = z_0$ is a *stagnation point* for the flow, then $w'(z_0) = 0$, by virtue of eqn (4.15). Unless $w''(z_0)$ also happens to be zero, it follows that the flow in the immediate neighbourhood of the stagnation point will be determined by the

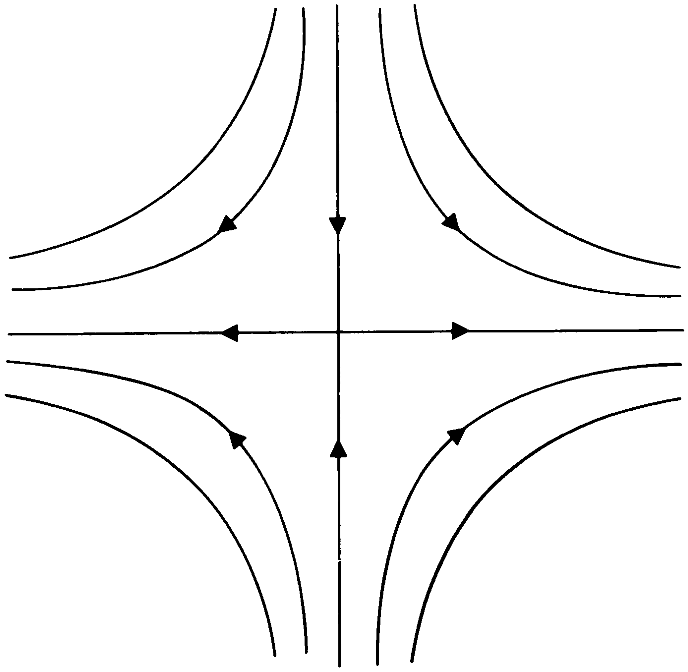


Fig. 4.1. 2-D irrotational flow near a stagnation point.

quadratic term in the above expression. Now, $w''(z_0)$ will typically be complex, $\alpha e^{i\beta}$, say, but by first shifting our coordinates:

$$z - z_0 = z_1,$$

so that the stagnation point is at $z_1 = 0$, and then rotating them so that

$$z_1 e^{i\beta/2} = z_2,$$

we may write

$$w = \text{constant} + \tfrac{1}{2}\alpha z_2^2 + \dots .$$

Dropping the inconsequential constant, we see that relative to suitably located and orientated coordinates the complex potential in the neighbourhood of a stagnation point is

$$w = \tfrac{1}{2}\alpha z^2, \tag{4.22}$$

where α is real, the corresponding flow being

$$u = \alpha x, \qquad v = -\alpha y \tag{4.23}$$

(cf. Exercise 1.7). The stream function is

$$\psi = \alpha xy, \tag{4.24}$$

so the streamlines are rectangular hyperbolae, as in Fig. 4.1.

4.4. The method of images

Suppose there is a line vortex of strength Γ at a distance d from a rigid plane wall $x = 0$, as in Fig. 4.2(*a*). A clever trick for obtaining the flow is to imagine that the region $x \leqslant 0$ is also filled with fluid and that there is an equal and opposite vortex, i.e. of strength $-\Gamma$, at the mirror-image point, as in Fig. 4.2(*b*). The reason for doing this is that the x-components of velocity of the two vortices obviously cancel on $x = 0$, so there is no normal velocity component there. Thus the complex potential

$$w = -\frac{\mathrm{i}\Gamma}{2\pi}\log(z - d) + \frac{\mathrm{i}\Gamma}{2\pi}\log(z + d) \tag{4.25}$$

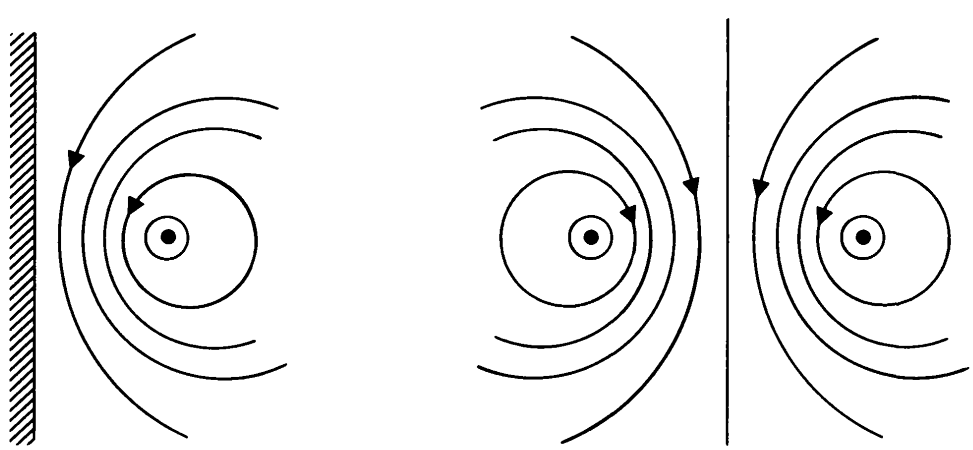
serves not only for the flow problem in Fig. 4.2(*b*) but, in $x \geqslant 0$, for the flow in the presence of a wall in Fig. 4.2(*a*). This is a simple example of the *method of images,* which is all about getting flows that satisfy boundary conditions.

Let us examine the flow in Fig. 4.2 a little more carefully. The stream function ψ is obtained by writing

$$\phi + \mathrm{i}\psi = -\frac{\mathrm{i}\Gamma}{2\pi}\log\left(\frac{z - d}{z + d}\right), \tag{4.26}$$

and the streamlines are therefore

$$\left|\frac{z - d}{z + d}\right| = \text{constant}. \tag{4.27}$$



(*a*) (*b*)

Fig. 4.2. Flows due to line vortices.

These are circles, the so-called coaxal circles of elementary geometry. Each circle cuts the circle $|z| = d$ orthogonally, and if the centre of any circle is distant c_1 and c_2 from the two vortices, then $c_1c_2 = a^2$, where a is its radius.

It is a simple matter, then, to write down the flow inside a circular cylinder $|z| = a$ due to a line vortex at $z = c < a$: it will be as if the cylinder were not present and there were, instead, an equal and opposite line vortex at $z = a^2/c$. The complex potential for the flow in Fig. 4.3 is therefore

$$w = -\frac{\mathrm{i}\Gamma}{2\pi}\log(z-c) + \frac{\mathrm{i}\Gamma}{2\pi}\log\left(z - \frac{a^2}{c}\right). \tag{4.28}$$

While it is not a matter of major concern at present, eqns (4.25) and (4.28) are, in fact, only *instantaneous* complex potentials corresponding to the momentary positions of the vortices; the vortices, and the whole streamline patterns associated with them, in fact move in a manner to be described in §5.6.

Milne-Thomson's circle theorem

Suppose we have a flow with complex potential $w = f(z)$, where all the singularities of $f(z)$ lie in $|z| > a$. Then

$$w = f(z) + \overline{f(a^2/\bar{z})}, \tag{4.29}$$

where an overbar denotes complex conjugate, is the complex potential of a flow with (i) the same singularities as $f(z)$ in $|z| > a$ and (ii) $|z| = a$ as a streamline.

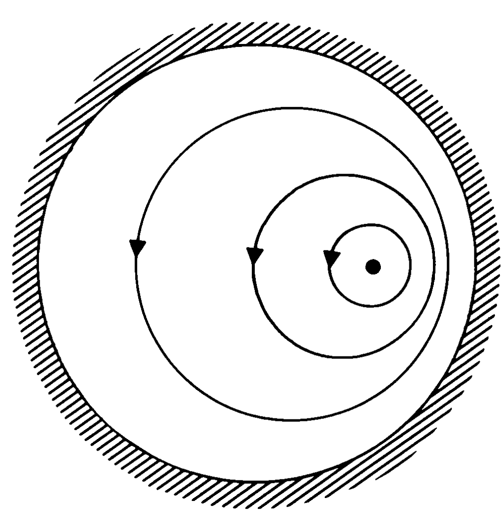


Fig. 4.3. Flow due to a line vortex inside a circular cylinder.

The last property makes the circle theorem a sort of automated method of images for circular boundaries. To prove it, note first that as all the singularities of $f(z)$ are in $|z|>a$, all those of $f(a^2/\bar{z})$ are in $|a^2/\bar{z}|>a$, i.e. in $|z|<a$. Second, on the circle itself we have $z\bar{z}=a^2$, so

$$w=f(z)+\overline{f(z)} \qquad \text{on } |z|=a. \tag{4.30}$$

Thus w is real on $|z|=a$, so $\psi=0$ there, so $|z|=a$ is a streamline.

An elementary application of the circle theorem follows in the next section.

4.5. Irrotational flow past a circular cylinder

Consider irrotational flow, uniform with speed U at infinity, past a fixed circular cylinder $|z|=a$. If the stream is parallel to the x-axis the complex potential for the undisturbed flow is $f(z)=Uz$, which has a singularity only at infinity. Applying the circle theorem we find

$$f(a^2/\bar{z})=Ua^2/\bar{z}, \qquad \overline{f(a^2/\bar{z})}=Ua^2/z,$$

so

$$w(z)=U\left(z+\frac{a^2}{z}\right) \tag{4.31}$$

is the complex potential of an irrotational flow, uniform at infinity, having $|z|=a$ as a streamline.

It is not the only irrotational flow satisfying these conditions; we may plainly superimpose a line vortex flow of arbitrary strength Γ to give

$$w(z)=U\left(z+\frac{a^2}{z}\right)-\frac{\mathrm{i}\Gamma}{2\pi}\log z \tag{4.32}$$

as the complex potential of a more general irrotational flow having no normal velocity at $|z|=a$, yet being uniform, with speed U, at infinity.

Nevertheless, consider first the case (4.31) in which there is no circulation round the cylinder. Putting $z=re^{\mathrm{i}\theta}$ we find that

$$\phi=U\left(r+\frac{a^2}{r}\right)\cos\theta \tag{4.33}$$

and

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta, \qquad (4.34)$$

whence†

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta. \qquad (4.35)$$

The flow is symmetric fore and aft of the cylinder, and some of the streamlines are sketched in Fig. 4.4(a).

There is evidently *slip* on the cylinder—according to this irrotational flow theory, at any rate—for

$$u_\theta = -2U\sin\theta \qquad \text{at } r = a. \qquad (4.36)$$

In discussing this it is convenient to use instead $u_s = -u_\theta$, which is positive, and $s = (\pi - \theta)a$, which is the distance along the top of the cylinder from the forward stagnation point. Thus

$$u_s = 2U\sin\frac{s}{a}, \qquad (4.37)$$

and

$$\frac{du_s}{ds} = \frac{2U}{a}\cos\frac{s}{a}.$$

The slip velocity therefore rises from zero at the front stagnation point to a maximum of $2U$ at $\theta = \pi/2$; it then decreases again to zero at the rear stagnation point.

When there is circulation Γ round the cylinder, as in eqn (4.32), the velocity components are

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}. \qquad (4.38)$$

Anticipating the applications to aerofoil theory that lie ahead, we have taken Γ to be negative in Fig. 4.4, so that the superimposed circulatory flow is clockwise. The character of the streamline

† We do not, of course, need the full apparatus of complex variable theory and circle theorem to establish this particular result; there is a much simpler way (Exercise 4.4).

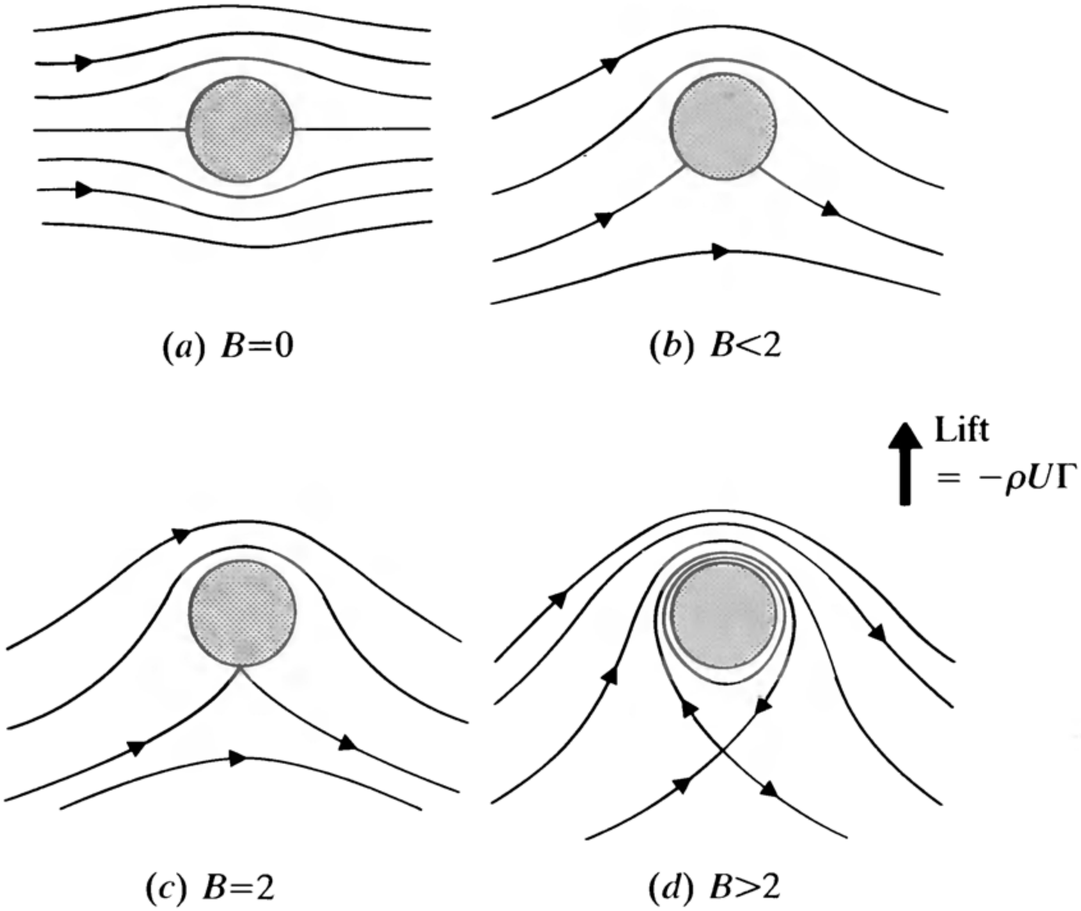


Fig. 4.4. Irrotational flows past a circular cylinder.

pattern depends crucially on the parameter

$$B = -\Gamma/2\pi U a, \tag{4.39}$$

which is then positive.

One notable feature of the flow that changes with B is the location of the stagnation points. When $B < 2$ there are two of them, both located on the cylinder $r = a$, at $\sin\theta = -\frac{1}{2}B$. They therefore move round as B is increased and coalesce when $B = 2$ at $\theta = 3\pi/2$. When $B > 2$ there is only one stagnation point, and it lies off the cylinder at

$$\frac{r}{a} = \frac{B}{2} + \left(\frac{B^2}{4} - 1\right)^{\frac{1}{2}}, \qquad \theta = \frac{3\pi}{2}. \tag{4.40}$$

This stagnation point thus moves further and further away from the cylinder as B increases, and the region of closed streamlines adjacent to the cylinder becomes steadily larger.

The net force on the cylinder may be calculated from the pressure distribution on $r = a$. As the cylinder is a streamline, and the motion is steady, Bernoulli's theorem gives

$$p + \tfrac{1}{2}\rho \mathbf{u}^2 = \text{constant} \qquad \text{on } r = a,$$

whence

$$\frac{p}{\rho} = \text{constant} - 2U^2 \sin^2\theta + \frac{U\Gamma}{\pi a} \sin\theta \qquad \text{on } r = a.$$

This pressure distribution is symmetric fore and aft of the cylinder (i.e. unchanged by the transformation $\theta \Rightarrow \pi - \theta$), so any net force must be perpendicular to the oncoming stream. The force on a small element $a\,\mathrm{d}\theta$ of the cylinder is $pa\,\mathrm{d}\theta$ (per unit length in the z-direction). The y-component of this force is $-pa \sin\theta\,\mathrm{d}\theta$, and there is therefore a net force on the cylinder of

$$\rho \int_0^{2\pi} \left(2U^2 \sin^2\theta - \frac{U\Gamma}{\pi a} \sin\theta\right) a \sin\theta\,\mathrm{d}\theta = -\rho U\Gamma \tag{4.41}$$

in the y-direction, in keeping with the far more general Kutta–Joukowski Lift Theorem of §4.11.

There is positive 'lift', then, if $\Gamma < 0$, and it is easy to see why this should be so, as we have already observed in §1.6. On top of the cylinder in Fig. 4.4 the circulatory flow reinforces the oncoming stream (if $\Gamma < 0$), leading to high speeds and low pressures. Beneath the cylinder the circulatory flow opposes the oncoming stream, leading to low speeds—as evinced by the stagnation points—and high pressures.

Before proceeding further we should emphasize again that we are currently using the irrotational flows in Fig. 4.4 purely as a mathematical device for the calculation of irrotational flows past a thin aerofoil. We are deferring, in particular, all question of whether the flows of Fig. 4.4 are *themselves* observable for a real (i.e. viscous) fluid, whether at high Reynolds number or otherwise (see §§5.7 and 8.6, cf. §7.7).

For what follows it is convenient, in fact, to take the oncoming stream at an angle α to the x-axis. The complex potential of the undisturbed flow is $Uz\mathrm{e}^{-\mathrm{i}\alpha}$, by virtue of eqn (4.17). Applying the circle theorem and superposing a line vortex flow of strength Γ then gives

$$w(z) = U\left(z\mathrm{e}^{-\mathrm{i}\alpha} + \frac{a^2}{z}\mathrm{e}^{\mathrm{i}\alpha}\right) - \frac{\mathrm{i}\Gamma}{2\pi} \log z \tag{4.42}$$

as our starting point, and this corresponds to the flows of Fig. 4.4 turned anticlockwise through an angle α.

4.6. Conformal mapping

Let $w(z)$ be the complex potential of some 2-D irrotational flow in the z-plane, with $w = \phi + i\psi$. Suppose now that we choose

$$Z = f(z) \tag{4.43}$$

as some analytic function of z, with an inverse

$$z = F(Z) \tag{4.44}$$

which is an analytic function of Z. Then

$$W(Z) = w\{F(Z)\} \tag{4.45}$$

is an analytic function of Z. Now write

$$Z = X + iY \tag{4.46}$$

and split $W(Z)$ into its real and imaginary parts:

$$W(Z) = \Phi(X, Y) + i\Psi(X, Y). \tag{4.47}$$

As W is an analytic function of Z, Φ and Ψ satisfy the Cauchy–Riemann equations, and it follows that the two functions

$$u_*(X, Y) = \partial\Phi/\partial X = \partial\Psi/\partial Y, \qquad v_*(X, Y) = \partial\Phi/\partial Y = -\partial\Psi/\partial X, \tag{4.48}$$

represent the velocity components of an irrotational, incompressible flow in the Z-plane.

Further, because $W(Z)$ and $w(z)$ take the same value at corresponding points of the two planes (i.e. points related by eqns (4.43) or (4.44)) it follows that Ψ and ψ are the same at corresponding points. Thus streamlines are mapped into streamlines. In particular, a fixed rigid boundary in the z-plane, which is necessarily a streamline, gets mapped into a streamline in the Z-plane, which could accordingly be viewed as a rigid boundary for the flow in the Z-plane. The key question, then, is: Given flow past a circular cylinder in the z-plane (see eqn (4.42)), can we choose the mapping (4.43) so as to obtain in the Z-plane uniform flow past a more wing-like shape?

What happens to the circulation round a closed circuit is important in this connection. Evidently Φ and ϕ are the same at corresponding points of the two planes, and it follows that if we go once round some closed circuit of the z-plane and obtain some consequent change in ϕ, we will obtain the same change in Φ on going once round the corresponding circuit in the Z-plane. Appealing to eqn (4.4), then, we see that the circulations round two such corresponding circuits must be the same.

What happens to the flow at infinity is also of importance. Plainly

$$\frac{\mathrm{d}W}{\mathrm{d}Z}=\frac{\mathrm{d}w/\mathrm{d}z}{\mathrm{d}Z/\mathrm{d}z}, \tag{4.49}$$

so

$$u_* - \mathrm{i}v_* = (u - \mathrm{i}v)/f'(z). \tag{4.50}$$

If we want to map uniform flow past some object into the same uniform flow past another object we must therefore choose $f(z)$ such that $f'(z) \to 1$ as $|z| \to \infty$.

One last general observation concerns a strictly local property of conformal mapping which gives the method its name. Take some point z_0 in the z-plane, with a corresponding point Z_0 in the Z-plane, and let $f^{(n)}(z_0)$ be the first non-vanishing derivative of the function $f(z)$ at z_0. Typically, n will be 1, but there will be occasions in what follows when $f'(z_0) = 0$ but $f''(z_0) \neq 0$, in which case $n = 2$. Let δz denote a small element in the z-plane, originating at $z = z_0$, and let δZ denote the corresponding element in the Z-plane, originating at $Z = Z_0$. By expanding $f(z)$ in a Taylor series we find that

$$\delta Z = \frac{(\delta z)^n}{n!} f^{(n)}(z_0) + O(\delta z)^{n+1}.$$

To first order in small quantities, then,

$$\arg(\delta Z) = n \arg(\delta z) + \arg\{f^{(n)}(z_0)\},$$

and it follows that if δz_1 and δz_2 denote two small elements in the z-plane, both originating at z_0, then

$$\arg(\delta Z_2) - \arg(\delta Z_1) = n[\arg(\delta z_2) - \arg(\delta z_1)]. \tag{4.51}$$

Thus when two short intersecting elements in the z-plane are mapped into two short intersecting elements in the Z-plane, the angle between them is multiplied by n. Usually, $n = 1$, and such angles are preserved. The shape of a small figure in the z-plane (e.g. a small parallelogram) is then preserved by the mapping—hence the name 'conformal'.

A very effective transformation for our purposes is the Joukowski transformation,

$$Z = z + \frac{c^2}{z}, \tag{4.52}$$

and we shall exploit the fact that $f'(\pm c) = 0$ but $f''(\pm c) \neq 0$, so that angles between two short line elements which intersect at either $z = c$ or $z = -c$ are doubled by the transformation. The inverse of eqn (4.52) is

$$z = \tfrac{1}{2}Z + (\tfrac{1}{4}Z^2 - c^2)^{\frac{1}{2}}, \tag{4.53}$$

although we have to take steps to pin down the meaning of this, for there are branch points at $Z = \pm 2c$. In all that follows we shall (i) cut the Z-plane along the real axis between $Z = -2c$ and $Z = 2c$, which stops eqn (4.53) from being multivalued, and (ii) interpret $(\frac{1}{4}Z^2 - c^2)^{\frac{1}{2}}$ as meaning that branch of the function which behaves like $\frac{1}{2}Z$ (as opposed to $-\frac{1}{2}Z$) as $|Z| \to \infty$, which ensures that $z \sim Z$ when $|Z|$ is large.

4.7. Irrotational flow past an elliptical cylinder

Consider the effect of the Joukowski transformation (4.52) on the circle $z = a\mathrm{e}^{\mathrm{i}\theta}$, where $0 \leqslant c \leqslant a$. Plainly

$$X + \mathrm{i}Y = \left(a + \frac{c^2}{a}\right)\cos\theta + \mathrm{i}\left(a - \frac{c^2}{a}\right)\sin\theta,$$

so the circle is mapped into the ellipse

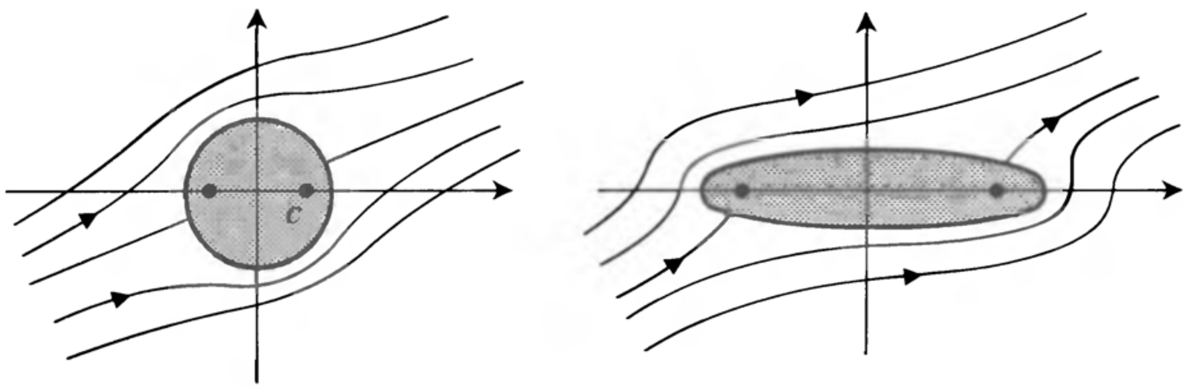
$$\frac{X^2}{(a + c^2/a)^2} + \frac{Y^2}{(a - c^2/a)^2} = 1 \tag{4.54}$$

in the Z-plane (see Fig. 4.5).

Substituting eqn (4.53) into eqn (4.42) we thus obtain

$$W(Z) = U\mathrm{e}^{-\mathrm{i}\alpha}[\tfrac{1}{2}Z + (\tfrac{1}{4}Z^2 - c^2)^{\frac{1}{2}}] + U\mathrm{e}^{\mathrm{i}\alpha}\frac{a^2}{c^2}[\tfrac{1}{2}Z - (\tfrac{1}{4}Z^2 - c^2)^{\frac{1}{2}}]$$

$$-\frac{\mathrm{i}\Gamma}{2\pi}\log[\tfrac{1}{2}Z + (\tfrac{1}{4}Z^2 - c^2)^{\frac{1}{2}}] \tag{4.55}$$



(*a*) z−plane (*b*) Z−plane

Fig. 4.5. Flow past an elliptical cylinder by conformal mapping; no circulation.

as the complex potential for uniform flow at an angle α past the ellipse (4.54), with circulation Γ. It is an elementary, but messy, exercise to write $Z = X + \mathrm{i}Y$ and then extract the imaginary part of $W(Z)$, namely $\Psi(X, Y)$. The streamlines are sketched in Fig. 4.5(*b*) for the case $\Gamma = 0$.

4.8. Irrotational flow past a finite flat plate

If we choose $c = a$, so that

$$Z = z + \frac{a^2}{z}, \tag{4.56}$$

the ellipse (4.54) collapses to a flat plate of length $4a$. Consider the velocity components u_* and v_* in the Z-plane:

$$u_* - \mathrm{i}v_* = \frac{\mathrm{d}W}{\mathrm{d}Z} = \frac{\mathrm{d}w/\mathrm{d}z}{\mathrm{d}Z/\mathrm{d}z} = \left(U\mathrm{e}^{-\mathrm{i}\alpha} - U\mathrm{e}^{\mathrm{i}\alpha}\frac{a^2}{z^2} - \frac{\mathrm{i}\Gamma}{2\pi z}\right)\bigg/\left(1 - \frac{a^2}{z^2}\right). \tag{4.57}$$

Using eqn (4.53) we can write them in terms of Z, but the comparative simplicity of eqn (4.57) can be more helpful for many purposes.

In particular, the flow speed is in general infinite at the ends of the plate ($Z = \pm 2a$), as these points correspond to the points $z = \pm a$. The status of these sharp edges as singular points in the flow is confirmed by a glance at the streamline pattern for the case $\Gamma = 0$ in Fig. 4.6(*a*).

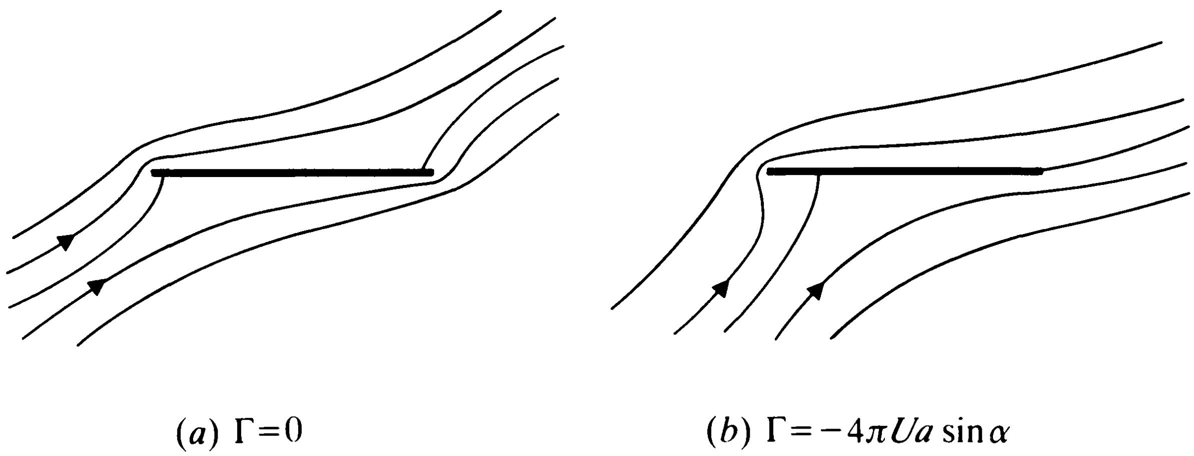


Fig. 4.6. Irrotational flow past a finite flat plate.

Notably, however, the singularity at the trailing edge $Z=2a$ (i.e. $z=a$) may be removed *if the circulation* Γ *is chosen so that the numerator in eqn* (4.57) *vanishes at the trailing edge.* Thus if

$$Ue^{-i\alpha}-Ue^{i\alpha}-\frac{i\Gamma}{2\pi a}=0,$$

i.e. if

$$\Gamma=-4\pi Ua\sin\alpha, \tag{4.58}$$

then by writing $z=a+\varepsilon$ in both the numerator and denominator of eqn (4.57) and taking the limit as $\varepsilon\to 0$ we find

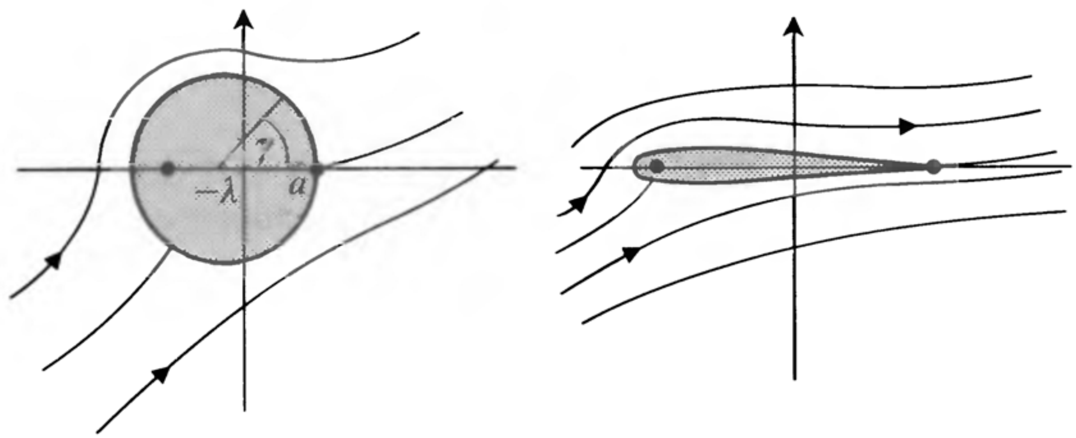
$$u_*\to U\cos\alpha,\quad v_*\to 0 \qquad \text{as } Z\to 2a,$$

so that the flow leaves the trailing edge smoothly and parallel to the plate, as in Fig. 4.6(*b*). The sense of the circulation is clockwise (for $\alpha>0$), and this is why we chose to represent the effects of a clockwise circulation in Fig. 4.4.

Of course, the presence of this circulation still leaves a singularity in the velocity field at the leading edge in Fig. 4.6(*b*).

4.9. Flow past a symmetric aerofoil

In view of Figs 4.5 and 4.6 it will come as no surprise that if we use the mapping (4.56) on a circle in the z-plane which passes through $z=a$ but which encloses $z=-a$, we obtain an aerofoil with a rounded nose but a sharp trailing edge, as in Fig. 4.7(*b*). If the centre of the circle is on the real axis in the z-plane, at $z=-\lambda$, say, the aerofoil is symmetric and given in terms of the



(*a*) z−plane (*b*) Z−plane

Fig. 4.7. Flow past a symmetric Joukowski aerofoil by conformal mapping.

parameter γ by

$$Z = -\lambda + (a+\lambda)e^{i\gamma} + \frac{a^2}{-\lambda + (a+\lambda)e^{i\gamma}}. \tag{4.59}$$

Its shape and thickness depend on λ.

The complex potential $W(Z)$ corresponding to uniform flow past this aerofoil at angle of attack α is obtained by first modifying eqn (4.42) to take account of the new radius and location of the cylinder in the z-plane:

$$w(z) = U\left[(z+\lambda)e^{-i\alpha} + \frac{(a+\lambda)^2}{(z+\lambda)}e^{i\alpha}\right] - \frac{i\Gamma}{2\pi}\log(z+\lambda),$$

and then substituting $z = \frac{1}{2}Z + (\frac{1}{4}Z^2 - a^2)^{\frac{1}{2}}$.

The counterpart to eqn (4.57) is

$$\frac{dW}{dZ} = \left\{U\left[e^{-i\alpha} - \left(\frac{a+\lambda}{z+\lambda}\right)^2 e^{i\alpha}\right] - \frac{i\Gamma}{2\pi(z+\lambda)}\right\}\Big/\left(1 - \frac{a^2}{z^2}\right), \tag{4.60}$$

but now it is only the vanishing of the denominator at $z = a$ ($Z = 2a$) that causes concern, for $z = -a$ corresponds to a point in the Z-plane which is inside the aerofoil. The value of Γ which makes the numerator in eqn (4.60) zero at the trailing edge ($z = a$) is

$$\Gamma = -4\pi U(a+\lambda)\sin\alpha. \tag{4.61}$$

The flow is then smooth and free of singularities everywhere, as shown in Fig. 4.7(*b*), and this is an example of the *Kutta–Joukowski condition* at work.

When $\lambda \ll a$ the aerofoil described by eqn (4.59) is thin and symmetric, with length approximately $4a$ and maximum thickness $3\sqrt{3}\lambda$. By neglecting λ in comparison with a in eqn (4.61) we obtain the classic expression (4.1).

4.10. The forces involved: Blasius's theorem

Let there be a steady flow with complex potential $w(z)$ about some fixed body which has as its boundary the closed contour C, as in Fig. 4.8. If F_x and F_y are the components of the net force (per unit length) on the body, then

$$F_x - \mathrm{i}F_y = \tfrac{1}{2}\mathrm{i}\rho \oint_C \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z. \tag{4.62}$$

This is *Blasius's theorem.*

To prove it, let s denote arc length along C, and let θ denote the angle made with the x-axis by the tangent to C. Then the force (per unit length) on a small element δs of the boundary is $(-\sin\theta, \cos\theta)p\,\delta s$, so

$$\delta F_x - \mathrm{i}\,\delta F_y = -p(\sin\theta + \mathrm{i}\cos\theta)\,\delta s = -p\mathrm{i}e^{-\mathrm{i}\theta}\,\delta s.$$

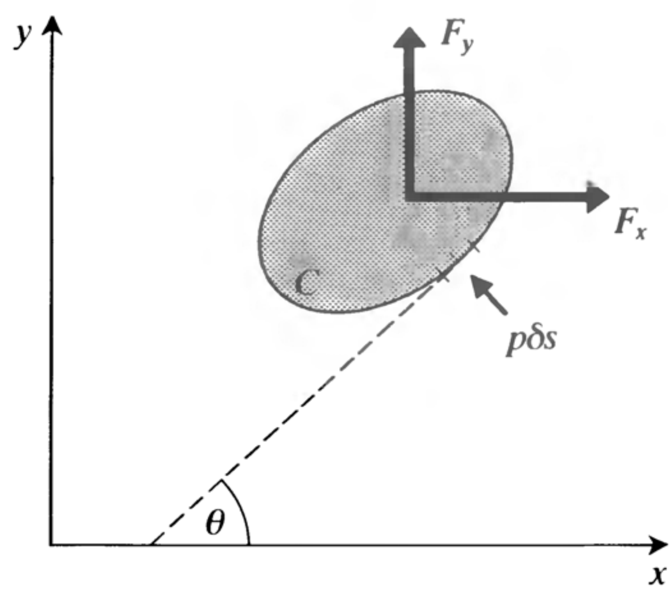


Fig. 4.8. Definition sketch for proof of Blasius's theorem.

Now, C is a streamline for the flow, so

$$u = q\cos\theta, \quad v = q\sin\theta \qquad \text{on } C,$$

where $q = (u^2 + v^2)^{\frac{1}{2}}$, so

$$\frac{dw}{dz} = u - iv = qe^{-i\theta} \text{ on } C.$$

Using Bernoulli's equation we may write

$$\delta F_x - i\,\delta F_y = (\tfrac{1}{2}\rho q^2 - k)ie^{-i\theta}\,\delta s,$$

where k is a constant, and substituting for q we find

$$\delta F_x - i\,\delta F_y = \tfrac{1}{2}i\rho\left(\frac{dw}{dz}\right)^2 e^{i\theta}\,\delta s - ki(\delta x - i\,\delta y).$$

Now, $e^{i\theta}\,\delta s = \delta z$. On integrating round the closed contour C the final term disappears and we obtain eqn (4.62).

In a similar way we may establish a formula for $\mathcal{N}$, the moment about the origin of the forces on the body:

$$\mathcal{N} = \text{Real part of}\left[-\tfrac{1}{2}\rho\oint_C z\left(\frac{dw}{dz}\right)^2 dz\right] \tag{4.63}$$

(see Exercise 4.5).

We now consider two examples.

Uniform flow past a circular cylinder

We have, of course, already calculated the net force in this case by direct integration of the pressure distribution in §4.5. Nevertheless, the complex potential is, in the case $\alpha = 0$:

$$w = U\left(z + \frac{a^2}{z}\right) - \frac{i\Gamma}{2\pi}\log z,$$

so applying Blasius's theorem:

$$F_x - iF_y = \tfrac{1}{2}i\rho\oint_C\left[U\left(1 - \frac{a^2}{z^2}\right) - \frac{i\Gamma}{2\pi z}\right]^2 dz.$$

When the integrand is expanded only the z^{-1} term gives a contribution to the integral. The coefficient of that term is

$-\mathrm{i}U\Gamma/\pi$, so a simple application of the residue calculus gives

$$F_x - \mathrm{i}F_y = \tfrac{1}{2}\mathrm{i}\rho \cdot 2\pi \mathrm{i} \cdot \left(-\frac{\mathrm{i}U\Gamma}{\pi}\right) = \mathrm{i}\rho U\Gamma.$$

Thus

$$F_x = 0, \qquad F_y = -\rho U\Gamma, \tag{4.64}$$

as found previously.

Uniform flow past an elliptical cylinder

Consider for simplicity the case when there is no circulation, as in Fig. 4.5(*b*). By the Kutta–Joukowski Lift Theorem (§4.11) there will be no net force on the ellipse, but there will in general be a torque about the origin given by eqn (4.63), i.e.

$$\text{Real part of}\left[-\tfrac{1}{2}\rho \oint_{\text{ellipse}} Z\left(\frac{\mathrm{d}W}{\mathrm{d}Z}\right)^2 \mathrm{d}Z\right].$$

Now, the expression (4.55) for W in terms of $Z = z + c^2/z$ is quite complicated, even in the case $\Gamma = 0$. It is more sensible, then, to write

$$\frac{\mathrm{d}W}{\mathrm{d}Z} = \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}Z}$$

and change the variable of integration from Z to z, so calculating

$$\text{Real part of}\left[-\tfrac{1}{2}\rho \oint_{|z|=a} Z\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \frac{\mathrm{d}z}{\mathrm{d}Z}\,\mathrm{d}z\right].$$

Now, when $\Gamma = 0$,

$$w = U\left(z\mathrm{e}^{-\mathrm{i}\alpha} + \frac{a^2}{z}\mathrm{e}^{\mathrm{i}\alpha}\right),$$

so the torque on the ellipse is the real part of

$$-\tfrac{1}{2}\rho U^2 \oint_{|z|=a} \left(z + \frac{c^2}{z}\right)\left(\mathrm{e}^{-\mathrm{i}\alpha} - \frac{a^2}{z^2}\mathrm{e}^{\mathrm{i}\alpha}\right)^2\left(1 - \frac{c^2}{z^2}\right)^{-1} \mathrm{d}z.$$

The integrand has poles at $-c$, 0, and c, all within the contour (as $0 < c < a$). Expanding the whole integrand in a Laurent series valid for $|z| > c$, and therefore valid on the integration contour, we obtain

$$\left(z+\frac{c^2}{z}\right)\left(e^{-2i\alpha}-\frac{2a^2}{z^2}+\frac{a^4}{z^4}e^{2i\alpha}\right)\left(1+\frac{c^2}{z^2}+\frac{c^4}{z^4}+\ldots\right).$$

The coefficient of z^{-1} is

$$c^2e^{-2i\alpha}-2a^2+c^2e^{-2i\alpha},$$

and the torque on the ellipse is therefore the real part of

$$-\tfrac{1}{2}\rho U^2\cdot 2\pi i\cdot(2c^2e^{-2i\alpha}-2a^2),$$

i.e.

$$\mathcal{N}=-2\pi\rho U^2c^2\sin 2\alpha. \tag{4.65}$$

For the flow in Fig. 4.5(*b*) the torque is negative, i.e. clockwise. More generally, it is such as to tend to align the ellipse so that it is broadside-on to the stream.

4.11. The Kutta–Joukowski Lift Theorem

Consider steady flow past a two-dimensional body, the cross-section of which is some simple closed curve C, as in Fig. 4.9. Let the flow be uniform at infinity, with speed U in the x-direction, and let the circulation round the body be Γ. Then

$$F_x=0,\qquad F_y=-\rho U\Gamma. \tag{4.66}$$

To prove this theorem, first choose the origin O so that it lies inside the body. Then, assuming the flow to be free of singularities, dw/dz will be an analytic function of z in the flow domain and can be expanded in a Laurent series valid for $R<|z|<\infty$, where R is the radius of the smallest circle centred on O which encloses the body. Furthermore, the form of this series must be

$$\frac{dw}{dz}=U+\frac{a_1}{z}+\frac{a_2}{z^2}+\ldots \tag{4.67}$$

because the flow is uniform, speed U, at infinity.

Now, we stated Blasius's theorem in the form of an integral (4.62) taken round the contour C of the body, but if the flow is free of singularities we may, by a cross-cut argument and use of Cauchy's theorem, take the integral equally well round any simple closed contour C' which surrounds the body. In

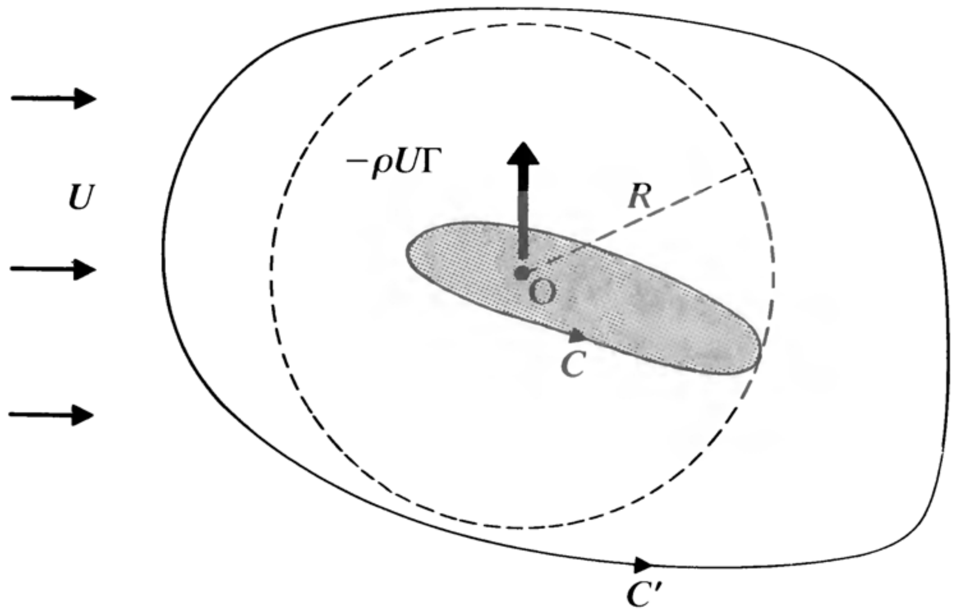


Fig. 4.9. Definition sketch for proof of the Kutta–Joukowski Lift Theorem.

particular, if we take it round a contour C', such as that in Fig. 4.9, which lies wholly in the region $|z|>R$, we may use eqn (4.67) to write

$$F_x - \mathrm{i}F_y = \tfrac{1}{2}\mathrm{i}\rho \oint_{C'} \left(U + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots\right)^2 \mathrm{d}z.$$

On expanding the integrand only the z^{-1} term contributes to the integral, and with residue $2Ua_1$ at $z=0$ this gives

$$F_x - \mathrm{i}F_y = \tfrac{1}{2}\mathrm{i}\rho \cdot 2\pi\mathrm{i} \cdot 2Ua_1 = -2\pi\rho U a_1. \tag{4.68}$$

To find a_1, use eqn (4.67) to write

$$2\pi\mathrm{i}a_1 = \oint_{C'} \frac{\mathrm{d}w}{\mathrm{d}z}\,\mathrm{d}z,$$

where C' lies wholly in $|z|>R$. We may then appeal again to Cauchy's theorem and a cross-cut argument to justify taking the integral round C instead of C', as $\mathrm{d}w/\mathrm{d}z$ is analytic in the whole of the flow region. Thus

$$2\pi\mathrm{i}a_1 = \oint_C \frac{\mathrm{d}w}{\mathrm{d}z}\,\mathrm{d}z = [w]_C = [\phi + \mathrm{i}\psi]_C.$$

But C is a streamline, so the change in ψ after one journey round C is zero. The change in ϕ, on the other hand, is simply Γ, the circulation round the body (see eqn (4.4)). Thus

$$2\pi i a_1 = \Gamma, \tag{4.69}$$

and substituting this in eqn (4.68) establishes the theorem, eqn (4.66).

4.12. Lift: the deflection of the airstream

Notwithstanding the importance of circulation, the Kutta–Joukowski condition, and the theorem of §4.11, an aerofoil obtains lift essentially by imparting downward momentum to the oncoming airstream. In the case of a single aerofoil in an infinite expanse of fluid this elementary truth is disguised, perhaps, by the way that the deflection of the airstream tends to zero at infinity. But in uniform flow past an infinite array of aerofoils, as in Fig. 4.10, there is a finite deflection of the airstream at infinity, so that the downward momentum flux is more readily apparent. Moreover, the deflection is related in a most instructive way to both the circulation and the lift. For this reason, it is worth exploring, and to do this we first need a reformulation of the equation of motion.

The steady momentum equation in integral form

For steady flow, and in the absence of body forces, Euler's equation (1.12) reduces to

$$\rho(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p,$$

and using a suffix notation and the summation convention this may be written

$$\rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i}.$$

Let us integrate this over some fixed region V which is enclosed by a fixed surface S, so that fluid is flowing in through some parts of S and out at others. Then the left-hand side becomes

$$\int_V \rho u_j \frac{\partial u_i}{\partial x_j} \, \mathrm{d}V = \int_V \rho \frac{\partial}{\partial x_j}(u_j u_i) \, \mathrm{d}V = \int_S \rho u_j u_i n_j \, \mathrm{d}S$$

$$= \int_S \rho(\boldsymbol{u} \cdot \boldsymbol{n}) u_i \, \mathrm{d}S,$$

the first equation holding because $\partial u_j/\partial x_j = \nabla \cdot \boldsymbol{u} = 0$, and the second holding by virtue of eqn (A.18). Thus

$$\int_S \rho(\boldsymbol{u}\cdot\boldsymbol{n})u_i \, \mathrm{d}S = -\int_V \frac{\partial p}{\partial x_i}\,\mathrm{d}V = -\int_S pn_i \, \mathrm{d}S,$$

where we have used eqn (A.15). In vector terms, then,

$$-\int_S p\boldsymbol{n}\,\mathrm{d}S = \int_S \rho\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{n})\,\mathrm{d}S. \tag{4.70}$$

Now, $\rho\boldsymbol{u}$ is the momentum per unit volume of a fluid element, and $(\boldsymbol{u}\cdot\boldsymbol{n})\,\delta S$ is the volume rate at which fluid is leaving a small portion δS of the surface S, so the right-hand side represents the rate at which momentum is getting carried out of S. The equation states, then, that the total force on S is equal to the rate at which momentum is carried out of S.

Flow past a stack of aerofoils

Let the (identical) aerofoils be a distance d apart, as in Fig. 4.10. Consider the flow in and out of the control surface ABCDA, where AB and DC are portions of identical streamlines a distance d apart, AD being far upstream, where the velocity is $(U, 0)$, and BC being far downstream, where we assume the velocity to be

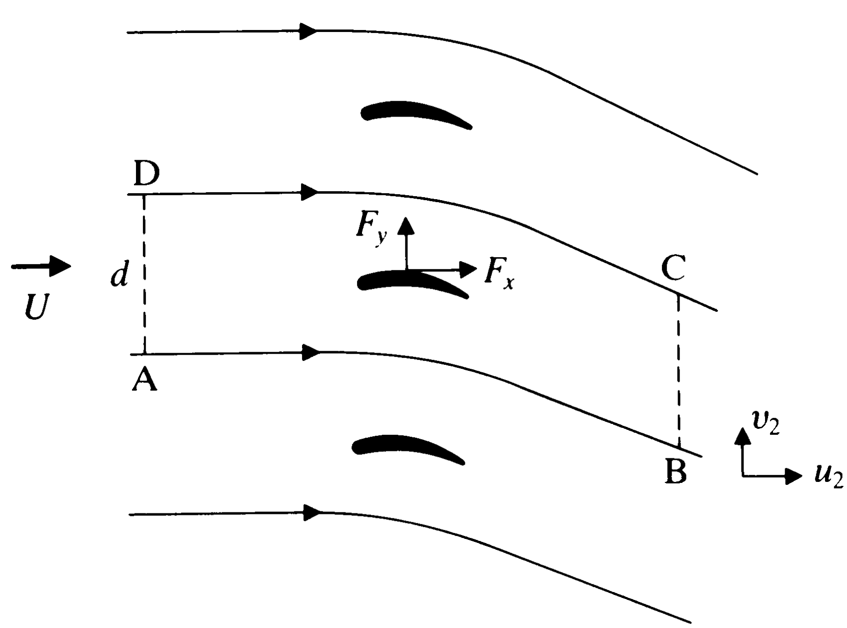


Fig. 4.10. Flow past a stack of aerofoils.

uniform again, but equal to (u_2, v_2). Now, because the fluid is incompressible the volume flux across AD must be equal to that across BC, so $Ud = u_2 d$, and therefore

$$u_2 = U. \tag{4.71}$$

We now apply the result (4.70) to the fixed region S which lies within ABCDA but excludes the aerofoil. If the lift on the aerofoil is F_y there is a vertical component of force $-F_y$ on S. (There is no other y-component to the first term in eqn (4.70), for those at BC and DA are zero and those at AB and CD cancel, because at any given x the pressures on AB and CD will be the same, as the flow repeats periodically in the y-direction.) There is no flux of momentum across either AB or CD, for they are streamlines, and there is no flux of vertical momentum across AD. Vertical momentum is, however, flowing out of BC at a rate $\rho v_2 Ud$ (per unit length in the z-direction). Equating this to the force exerted on S by the aerofoil, we have

$$F_y = -\rho U v_2 d. \tag{4.72}$$

In this way we see clearly how the lift is related to the deflection of the airstream; a downward deflection $(v_2 < 0)$ corresponds to positive lift. Moreover, it is clear, too, how the circulation is related to this deflection, and hence to the lift itself, for the circulation round ABCDA is

$$\Gamma = v_2 d, \tag{4.73}$$

as the contribution from DA is zero and those from AB and CD cancel. Thus

$$F_y = -\rho U \Gamma, \tag{4.74}$$

so that the Kutta–Joukowski result for a single aerofoil in fact holds in this rather different situation also.

4.13. D'Alembert's paradox

Consider the steady flow of an ideal fluid around a 3-D body which is placed in a long straight channel of uniform cross-section (Fig. 4.11). Let us apply eqn (4.70) to the fixed region bounded by the obstacle, two fixed cross-sections S_1 and S_2, and the channel walls. The net force in the downstream direction on the

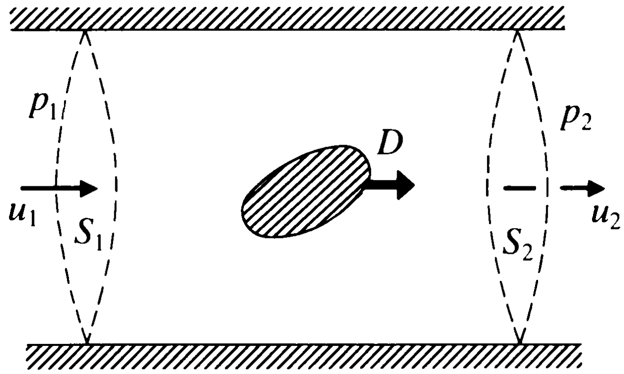


Fig. 4.11. Definition sketch for D'Alembert's paradox.

boundary of this region is

$$\int_{S_1} p_1 \, dS - \int_{S_2} p_2 \, dS - D,$$

where D is the drag exerted by the fluid on the obstacle. According to eqn (4.70), this net force is equal to the downstream component of the flux of momentum out of the region, which is

$$\rho \int_{S_2} u_2^2 \, dS - \rho \int_{S_1} u_1^2 \, dS,$$

where u_1 and u_2 are the velocity components parallel to the channel walls at S_1 and S_2. Thus

$$D = \int_{S_1} (p_1 + \rho u_1^2) \, dS - \int_{S_2} (p_2 + \rho u_2^2) \, dS. \qquad (4.75)$$

Now let us assume that the flow is uniform with speed U_0 far upstream, so that the pressure is a constant, p_0, there. Let us assume that conditions far downstream are similarly uniform; then considerations of mass flow show that the speed must again be U_0 far downstream, as the cross-sectional area of the channel has not changed. Applying the Bernoulli streamline theorem (1.16) to a streamline that runs along the channel walls from $x = -\infty$ to $x = +\infty$ we find that the uniform pressure far downstream must again be p_0.

If, then, we let the cross-sections S_1 and S_2 in Fig. 4.11 recede to infinity in the upstream and downstream directions, we see that the two competing integrals in eqn (4.75) tend to the same limit, and we therefore deduce that

$$D = 0. \tag{4.76}$$

This is one of several ways of presenting *D'Alembert's paradox*, namely that steady, uniform flow of an ideal fluid past a fixed body gives no drag on the body.

Another instructive way of viewing this result is as follows. Consider a finite rigid body which has as its boundary a simple closed surface S, and suppose that it is immersed in an infinite expanse of ideal fluid, the entire system being initially at rest. Suppose that the body now moves with speed $U(t)$ in the negative x-direction. The resulting flow is necessarily irrotational (§5.2), and it is, at any instant, unique (Exercise 5.24), determined entirely by the instantaneous normal component of velocity at the surface of the body. Indeed, at any instant the kinetic energy $T(t)$ of the fluid is proportional to the square of $U(t)$, the constant of proportionality being simply a function of the shape and size of the body (see, e.g., Exercise 5.27). Now, if D is the drag exerted on the body (i.e. the force opposite to the direction of $U(t)$), then the rate at which the fluid does work on the body is $-DU$. Equivalently, the body does work on the fluid at a rate DU, and the only way this energy can appear, in the present circumstances,† is as the kinetic energy of the fluid. So

$$DU = \mathrm{d}T/\mathrm{d}t. \tag{4.77}$$

There is therefore a drag on the body during the starting process, because the body needs to do work to set up all the kinetic energy of the fluid. But suppose that after a certain time the translational velocity U is held constant. D is then zero, according to eqn (4.77), because the kinetic energy of the fluid remains constant (although it is redistributed, of course, in a rather trivial way, as the whole streamline pattern shifts to follow the body).

The above energy argument can be adapted quite easily for 2-D flow past a 2-D object, provided that there is no circulation; if there is circulation round the object the kinetic energy T is typically infinite, and the argument based on eqn (4.77) breaks

† Equation (4.77) does not hold for a viscous fluid, because this energy can then be dissipated (§6.5). Nor does it hold when water waves or sound waves are present, because they can radiate energy to infinity (see, e.g., §3.7).

down. The result nevertheless obtains; according to the Kutta–Joukowski Lift Theorem (4.66) the drag is zero, whether or not there is any circulation.

The result flies in the face of common experience; bodies moving through a fluid are usually subject to a substantial resistance, or drag. In Fig. 4.12 we see the drag on a circular cylinder plotted as a function of the Reynolds number, and it remains substantial even when R is changed from 10^2 to 10^7, which is equivalent to decreasing the viscosity by five orders of magnitude. But then, as the sketches indicate, the flow as a whole shows no sign of settling down to the form in Fig. 4.4(*a*) as $\nu \to 0$. This is because the mainstream flow speed would, in that event, decrease very substantially along the boundary at the rear of the cylinder, and there would therefore be a strong adverse pressure gradient. An attached boundary layer cannot cope with that (see §2.1), and separation of the boundary layer leads instead to a substantial *wake* behind the cylinder. This wake changes in character with increasing R, as in Fig. 4.12, but shows no sign of disappearing as $R \to \infty$.

D'Alembert described his result of zero drag as 'a singular paradox'. His original argument (*c.* 1745) was in fact quite different to any of those above, and applied only to flow past

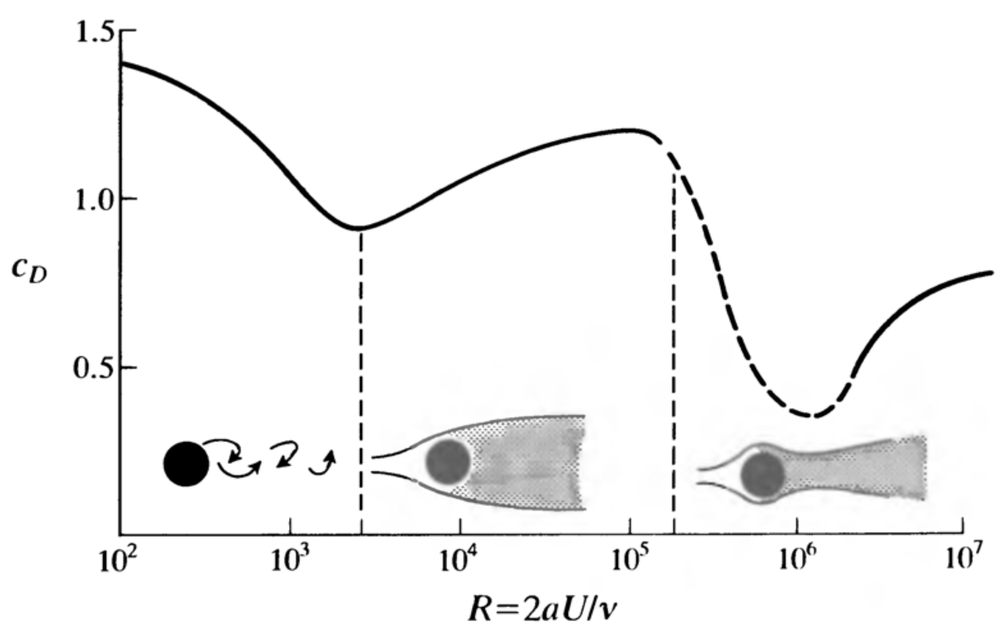


Fig. 4.12. Drag coefficient $c_D = D/\rho U^2 a$ for flow past a circular cylinder of radius *a*.

bodies, such as a sphere, that have fore–aft symmetry (see Exercise 5.26). Such an appeal to symmetry is unnecessary, and Euler came across the full 'paradox' quite independently. His argument involved consideration of the balance of momentum, but it differed significantly from the first argument presented above, not least because the concept of internal pressure p was not secure at the time (see §6.1).

Lighthill (1986) argues that 'D'Alembert's paradox' might better be designated 'D'Alembert's theorem', for if only a body is designed so as to avoid the kind of boundary layer separation evident in Fig. 4.12, then very low drag forces may indeed be achieved. The key feature in this respect is a long, slowly tapering rear to the body—as with an aerofoil—for this typically implies a very *weak* adverse pressure gradient at the rear of the body, enabling the boundary layer to remain attached. For flow past such a 'streamlined' body c_D is typically $O(R^{-\frac{1}{2}})$ as $R \to \infty$ (see eqn (8.24)).

Exercises

4.1. (i) Show that in a simply connected region of irrotational fluid motion the integral (4.2) is independent of the path between O and P.

(ii) Show that in a simply connected region of two-dimensional, incompressible fluid motion the integral

$$\psi = \int_O^P u \, dy - v \, dx$$

is independent of the path between O and P, and hence serves as a definition of the stream function.

4.2. The velocity field

$$u_r = \frac{Q}{2\pi r}, \qquad u_\theta = 0,$$

where Q is a constant, is called a *line source* flow if $Q > 0$ and a *line sink* if $Q < 0$. Show that it is irrotational and that it satisfies $\nabla \cdot \boldsymbol{u} = 0$, save at $r = 0$, where it is not defined. Find the velocity potential and the stream function, and show that the complex potential is

$$w = \frac{Q}{2\pi} \log z.$$

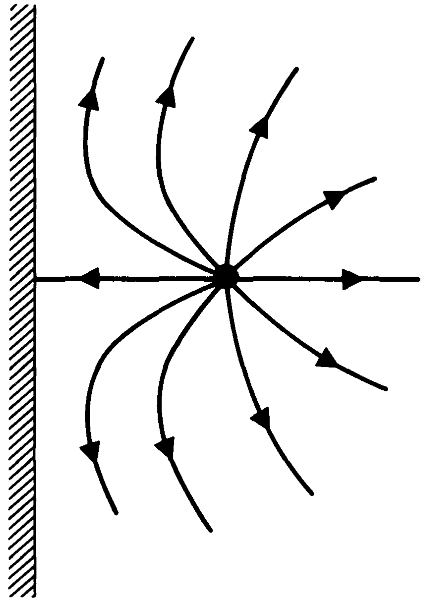


Fig. 4.13. Irrotational flow due to a line source near a wall.

Observe that the stream function is a multivalued function of position. Why does this not contradict part (ii) of Exercise 4.1?

Fluid occupies the region $x \geqslant 0$, and there is a plane rigid boundary at $x = 0$. Find the complex potential for the flow due to a line source at $z = d > 0$, and show that the pressure at $x = 0$ decreases to a minimum at $|y| = d$ and thereafter increases with $|y|$.

[Any attempt to reproduce the flow of Fig. 4.13 at high Reynolds number would be fraught with difficulties. A viscous boundary layer would be present, to satisfy the no-slip condition, but for $|y| > d$ the substantial adverse pressure gradient along the boundary would make separation inevitable (see §2.1). More fundamentally still, there are considerable practical difficulties in producing a line source, as opposed to a line sink, at high Reynolds number. These are more easily seen by considering the corresponding 3-D problem; a point sink can be simulated quite well by sucking at a small tube inserted in the fluid, but blowing down such a tube produces not a point source but a highly directional and usually turbulent jet (see, e.g. Lighthill 1986, pp. 100–103). The streamline pattern in Fig. 4.13 may nevertheless be observed in a *Hele–Shaw cell* (§7.7), although viscous effects are then paramount throughout the whole flow, so the pressure distribution is not given by Bernoulli's equation.]

4.3. An irrotational 2-D flow has stream function $\psi = A(x - c)y$, where A and c are constants. A circular cylinder of radius a is introduced, its centre being at the origin. Find the complex potential, and hence the stream function, of the resulting flow. Use Blasius's theorem (4.62) to calculate the force exerted on the cylinder.

4.4. Show that the problem of irrotational flow past a circular cylinder may be formulated in terms of the velocity potential $\phi(r, \theta)$ as follows:

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0,$$

with

$$\phi \sim Ur\cos\theta \quad \text{as } r\to\infty, \qquad \partial\phi/\partial r=0 \quad \text{on } r=a,$$

and obtain the solution (4.33) by using the method of separation of variables.

When there is circulation round the cylinder, derive eqn (4.40), and confirm that the stagnation points vary in position with the parameter B in the manner of Fig. 4.4.

4.5. Establish the expression (4.63) for the moment, $\mathcal{N}$, of forces on a body in irrotational flow, using an argument similar to that for Blasius's theorem.

4.6. By writing $z = a + \varepsilon$ in eqn (4.57) and taking the limit $\varepsilon \to 0$ check that the choice of circulation (4.58) does indeed lead to a finite velocity at the trailing edge.

4.7. According to eqns (4.1) and (4.66), the force on a thin symmetric aerofoil with a sharp trailing edge is

$$\mathcal{L} = \pi\rho U^2 L \sin\alpha$$

in a direction *perpendicular to the uniform stream.* This amounts to a component $\mathcal{L}\cos\alpha$ perpendicular to the aerofoil and a component $\mathcal{L}\sin\alpha$ parallel to the aerofoil, directed towards the leading edge. This latter component is, at first sight, rather curious; it might be thought that the net effect of a pressure distribution on a thin symmetric aerofoil should be almost normal to the aerofoil. That it is not is due to *leading edge suction*, i.e. a severe drop in pressure in the immediate vicinity of the rounded leading edge, this pressure drop being sufficient to make itself felt despite the small thickness of the wing on which it acts.

To see evidence of this, consider the extreme case of flow past a flat plate with circulation, as in Fig. 4.6(*b*) or Fig. 4.15. First, use eqns (4.56) and (4.57), on $z = ae^{i\theta}$, with Γ chosen according to eqn (4.58), to show that the flow speed on the plate is

$$U\left|\cos\alpha \pm \left(\frac{1-s}{1+s}\right)^{\frac{1}{2}}\sin\alpha\right|,$$

where the upper/lower sign corresponds to the upper/lower side of the plate, and s denotes $X/2a$, which therefore runs between -1 at the leading edge and $+1$ at the trailing edge.

Show that the corresponding pressure distributions are

$$p(s) = p(1) - \tfrac{1}{2}\rho U^2\left[\left(\frac{1-s}{1+s}\right)\sin^2\alpha \pm 2\left(\frac{1-s}{1+s}\right)^{\frac{1}{2}}\sin\alpha\cos\alpha\right],$$

(see Fig. 4.14). Note that there is a (negative) pressure singularity at the leading edge, whereas if the leading edge were rounded this pressure drop would be finite.

As far as the force component normal to the plate is concerned, note that the pressure difference across the plate is

$$p_D = 2\rho U^2\left(\frac{1-s}{1+s}\right)^{\frac{1}{2}}\sin\alpha\cos\alpha.$$

This too has a singularity at the leading edge, but it is integrable. Show that

$$\int_{-2a}^{2a} p_D\,\mathrm{d}X = \mathscr{L}\cos\alpha,$$

in keeping with the Kutta–Joukowski Lift Theorem.

Finally, show that eqn (4.65) holds even if there is circulation Γ round the ellipse, and then take the case $c = a$ to show that the torque on a flat plate about the origin is $-\mathscr{L}a\cos\alpha$, i.e. as if the whole lift force $\mathscr{L}$ were

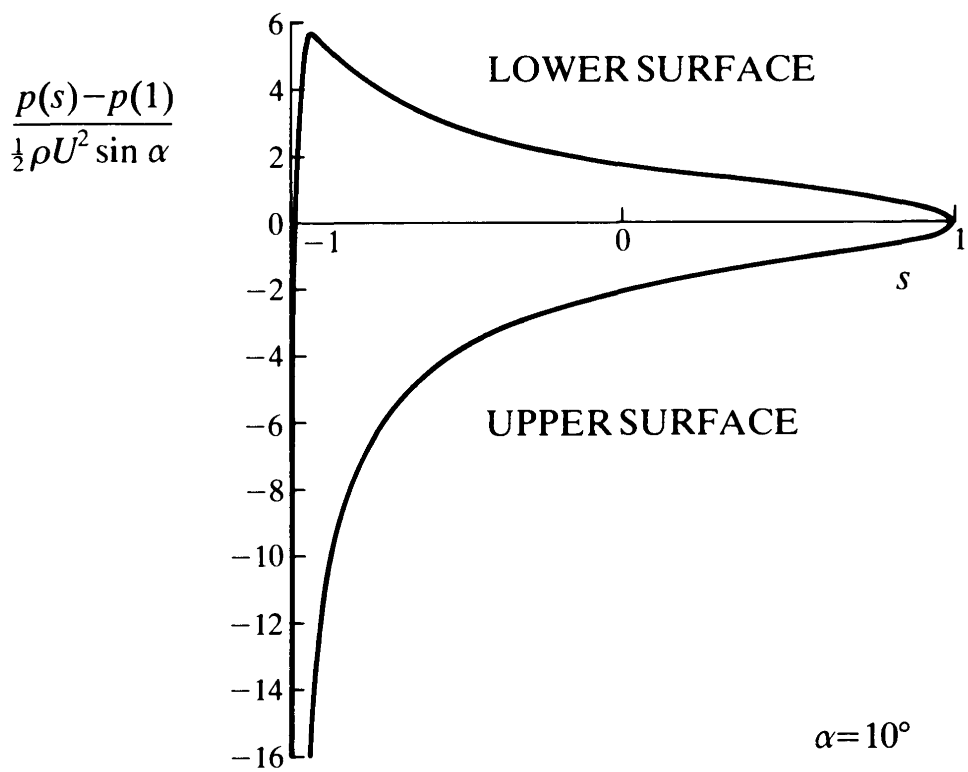


Fig. 4.14. Theoretical pressure distribution on a flat plate at a 10° angle of attack.

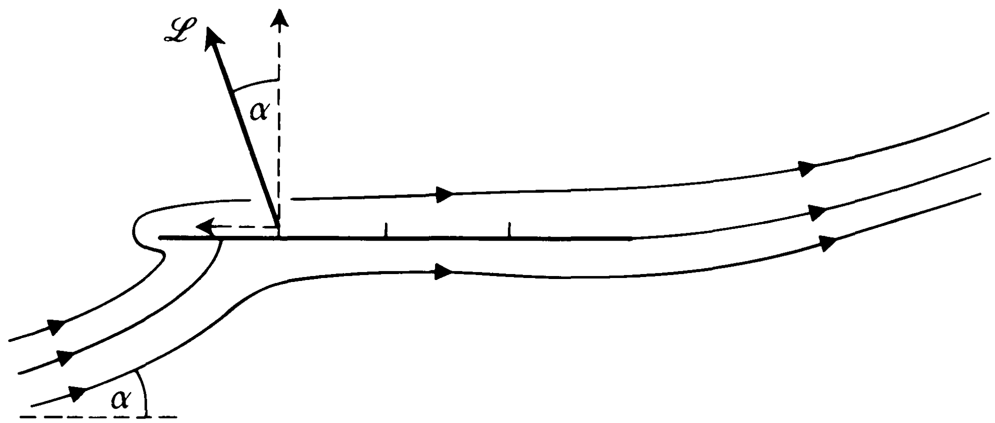


Fig. 4.15. The torque on a flat plate in uniform flow is as if the lift $\mathscr{L}$ were concentrated at a point one-quarter of the way along the plate from the leading edge.

applied at a point one-quarter of the way along the plate, as indicated in Fig. 4.15.

[The fact that this point is independent of α is of practical value, and makes for smooth control of an aircraft.]

4.8. Show that the Joukowski transformation $Z = z + a^2/z$ can be written in the form

$$\frac{Z - 2a}{Z + 2a} = \left(\frac{z - a}{z + a}\right)^2,$$

so that, in particular,

$$\arg(Z - 2a) - \arg(Z + 2a) = 2[\arg(z - a) - \arg(z + a)].$$

Consider the circle in the z-plane which passes through $z = -a$ and $z = a$ and has centre $\mathrm{i}a \cot \beta$. Show that the above transformation takes it into a circular arc between $Z = -2a$ and $Z = 2a$, with subtended angle 2β (Fig. 4.16). Obtain an expression for the complex potential in the Z-plane, when the flow is uniform, speed U, and parallel to the real axis. Show that the velocity will be finite at both the leading and trailing edges if

$$\Gamma = -4\pi Ua \cot \beta.$$

[This exceptional circumstance arises only when the undisturbed flow is parallel to the chord line of the arc.]

4.9. Provided that $f'(z_0) \neq 0$, points in the neighbourhood of $z = z_0$ are mapped by $Z = f(z)$, according to Taylor's theorem, in such a way that

$$Z - Z_0 = f'(z_0)(z - z_0) + O(z - z_0)^2,$$

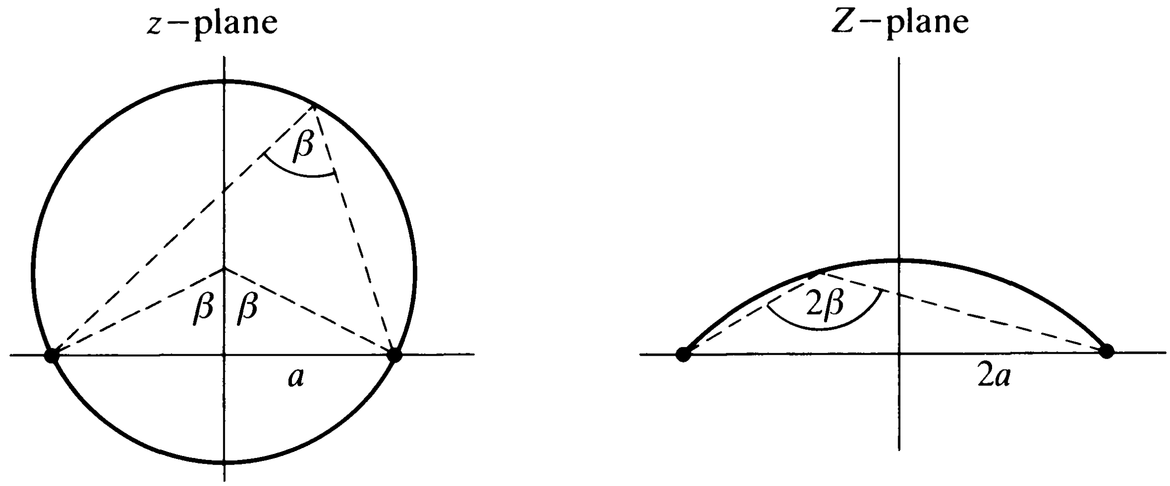


Fig. 4.16. Generation of a circular arc by a Joukowski transformation.

where $Z_0 = f(z_0)$. Use this to show that a line source of strength Q at $z = z_0$ is mapped into a line source of strength Q at $Z = Z_0$, provided that $f'(z_0) \neq 0$.

Fluid occupies the region between two plane rigid boundaries at $y = \pm b$, and there is a line source of strength Q at $z = 0$. Find the complex potential $w(z)$ for the flow

(i) by the method of images,

(ii) by using the mapping $Z = e^{\alpha z}$ with a suitably chosen $\alpha > 0$.

4.10. Use the momentum equation in its integral form (4.70) to show that there is a non-zero drag

$$F_x = \rho \Gamma^2 / 2d$$

on each of the aerofoils in Fig. 4.10.

Is this at odds with the Kutta–Joukowski Lift Theorem (4.66)?