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Elementary Fluid Dynamics

CLARENDON PRESS · OXFORD

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.
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New Delhi Shanghai Taipei Toronto

With offices in

Argentina Austria Brazil Chile Czech Republic France Greece
Guatemala Hungary Italy Japan South Korea Poland Portugal
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Published in the United States
by Oxford University Press Inc., New York

First published 1990
Reprinted 1990, 1992, 1994
1995, 1996, 1998, 2000, 2001, 2003 (twice, once with correction), 2005

A catalogue record for this book is available from the British Library

Library of Congress Cataloging in Publication Data
Acheson, D. J.
Elementary, fluid dynamics/D. J. Acheson.
p. cm.—(Oxford applied mathematics and computing science series)
1. Fluid dynamics. I. Title. II. Series.
TA357.A276 1990
532—dc20 89–22947 CIP

ISBN 0 19 859679 0 (Pbk)

Printed in Great Britain
on acid-free paper by
Biddles Ltd., King's Lynn, Norfolk

Preface

This book is an introduction to fluid dynamics for students of applied mathematics, physics, and engineering. The main mathematical requirements are the vector calculus and simple methods for solving differential equations. Exercises are provided at the end of each chapter, and extensive hints and answers are offered at the end of the book. In order to indicate how the text is organized it is first necessary to say a little about the subject itself.

It is a matter of common experience that some fluids are more *viscous* than others. No reader will be surprised to learn that the 'coefficient of viscosity' μ is much greater for syrup than it is for water. Many fluids, such as water and air, hardly seem to be viscous at all. It is natural, then, to construct a theory based on the concept of an *inviscid* fluid, i.e. one for which μ is precisely zero. This is how the subject first developed, and this is how we begin, in Chapter 1.

Yet inviscid theory has its dangers. Careful analysis of the equations of motion for a viscous fluid shows that strange things can happen in the limit $\mu \to 0$, so that *a fluid with very small viscosity may behave quite differently to a (hypothetical) fluid with no viscosity at all.* For this reason an elementary account of viscous flow appears very early in the book, in Chapter 2. The aim there, particularly in §§ 2.1 and 2.2, is to introduce some of the key ideas as simply as possible. In order to do this the viscous flow equations are merely stated; their derivation from first principles appears later.

While inviscid theory has to be used with caution there are major areas of fluid dynamics in which it is extremely successful, and one of these is wave motion (Chapter 3). Another is flow past a thin wing (Chapter 4), provided that the wing makes only a small angle of incidence with the oncoming stream. Inviscid theory has a further role in the study of vortex motion (Chapter 5), which turns out to be central to much of fluid dynamics, largely through the elegant theorems of Kelvin and Helmholtz.

In Chapter 6 we establish the equations of viscous flow from first principles, although some readers may wish to consult this chapter quite early. In Chapter 7 we explore very viscous flow, i.e. the case in which μ is large (in some appropriate sense). The flow problems here have some novel features and are the object of much current research. We return to fluids of low viscosity in Chapter 8, focusing on thin 'boundary layers', where viscous effects are of crucial importance, no matter how small μ happens to be. In the final chapter we examine the instability of fluid flow, which, together with boundary layer separation, gives rise to some of the deepest and most challenging problems in the subject.

I am extremely grateful to all the students who have tried out successive drafts of this book. I would also like to thank Brooke Benjamin, David Crighton, Raymond Hide, Tom Mullin, Hilary Ockendon, John Ockendon, Norman Riley, John Roe, Alan Tayler, and Robert Terrill for their comments on various chapters.

Finally, I take the opportunity to acknowledge all the help I received, when I was first learning the subject, from Raymond Hide at the Meteorological Office and from Norman Riley, Michael Glauert, and others at the University of East Anglia.

Jesus College, Oxford
April 1989 D. J. A.

Contents

1 Introduction

1.1. An experiment

Take a shallow dish and pour in salty water to a depth of 1 cm. Make a model wing with a length and span of 2 cm or so, ensuring that it has a sharp trailing edge. (One method is to cut the wing out of an india rubber with a knife.) Dip the wing vertically in the water and turn it to make a small angle of attack α with the direction in which it is to be moved. Put a blob of ink or food colouring around the trailing edge; a thin layer of this should then float on the salt water.

Now move the wing across the dish, giving it a clean, sudden start. If α is not too large there should be a strong anticlockwise vortex left behind at the point where the trailing edge started, as in Fig. 1.1.

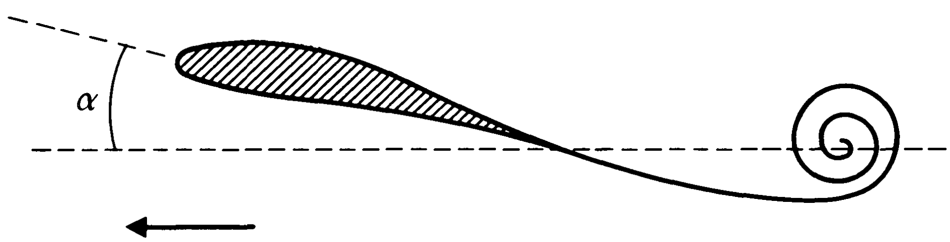


Fig. 1.1. The starting vortex.

A 'starting vortex' of this kind forms a crucial part of the mechanism by which an aircraft obtains lift, and we shall use aerodynamics in this chapter as a means of introducing some fundamental concepts of fluid flow.

Aerodynamics is, arguably, well suited to this purpose, but it goes without saying that the theory of fluid motion finds application in a wide variety of different fields. Within this book alone we may point to waves on a pond (§3.1), the instability of flow down a pipe (§9.1), the hydraulic jump in a kitchen sink

(§3.10), the interaction of two smoke rings (§5.4), the jet stream in the atmosphere (§9.8), the motion of quantum vortices in liquid helium (§5.8), the flow of volcanic lava (§7.9), the swimming of biological micro-organisms (§7.5), and the spin-down of a stirred cup of tea (§8.5) as examples of the breadth and diversity of the subject.

1.2. Some preliminary ideas

The usual way of describing a fluid flow is by means of an expression

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) \tag{1.1}$$

for the flow velocity $\boldsymbol{u}$ at any point $\boldsymbol{x}$ and at any time t. This tells us what all elements of the fluid are doing at any time; finding eqn (1.1) is usually the main task.

In general we must expect this task to be quite difficult. Let us take Cartesian coordinates, for example, and denote the three components of $\boldsymbol{u}$ by u, v, and w. Then eqn (1.1) is a convenient shorthand for

$$u = u(x, y, z, t), \qquad v = v(x, y, z, t), \qquad w = w(x, y, z, t).$$

There are, however, special classes of flow which have simplifying features.

A *steady* flow is one for which

$$\frac{\partial \boldsymbol{u}}{\partial t} = 0, \tag{1.2}$$

so that $\boldsymbol{u}$ depends on $\boldsymbol{x}$ alone. At any fixed point in space the speed and direction of flow are both constant.

A *two-dimensional* (2-*D*) *flow* is of the form

$$\boldsymbol{u} = [u(x, y, t), v(x, y, t), 0], \tag{1.3}$$

so that $\boldsymbol{u}$ is independent of one spatial coordinate (here selected to be z) and has no component in that direction.

A *two-dimensional steady flow* is thus of the form

$$\boldsymbol{u} = [u(x, y), v(x, y), 0]. \tag{1.4}$$

These are idealizations. No real flow can be exactly two-dimensional, but in the case of flow past a fixed wing of long span and uniform cross-section we might reasonably expect a close approximation to 2-D flow, except near the wing-tips.

Before exploring such a flow more closely it is useful to introduce the concept of a *streamline*. This is, at any particular time t, a curve which has the same direction as $\boldsymbol{u}(\boldsymbol{x}, t)$ at each point. Mathematically, then, a streamline $x = x(s)$, $y = y(s)$, $z = z(s)$ is obtained by solving

$$\frac{\mathrm{d}x/\mathrm{d}s}{u} = \frac{\mathrm{d}y/\mathrm{d}s}{v} = \frac{\mathrm{d}z/\mathrm{d}s}{w} \tag{1.5}$$

at a particular time t.

To imagine streamlines it can be convenient to consider a widely used experimental technique which involves putting tiny, neutrally buoyant polystyrene beads into the fluid. One particular plane of the fluid region is then illuminated by a collimated light beam, and the beads reflect this light to the camera, thus appearing as tiny pin-pricks of light if they are stationary. When the fluid is moving, however, the beads get carried around with it, so that a short-exposure-time photograph consists of short streaks, the length and direction of each one giving a measure of the fluid velocity at that particular point in space. As an example, we show in Fig. 1.2 a streak photograph for the flow (with uniform velocity at infinity) past a fixed wing. Because this is a steady flow the streamline pattern is the same at all times, and a fluid particle started on some streamline will travel along that

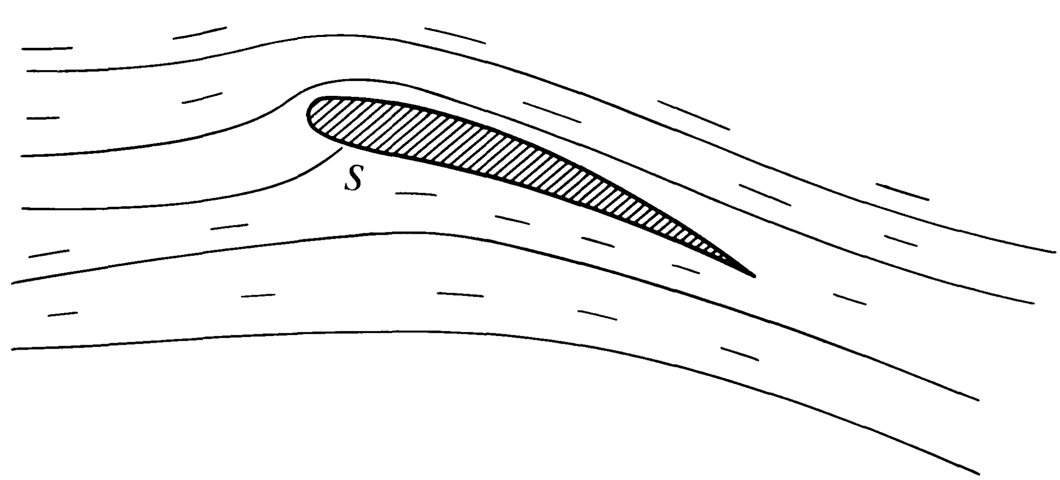


Fig. 1.2. Streamlines for steady flow past a fixed wing, as inferred from a streak photograph.

streamline as time proceeds. (In an unsteady flow, on the other hand, streamlines and particle paths are usually quite different; see Exercise 1.8.)

It is evident from Fig. 1.2 that even though the flow is steady, so that $\boldsymbol{u}$ is constant at a point fixed in space, $\boldsymbol{u}$ changes *as we follow any particular fluid element.* In particular—changes in direction of flow aside—an element riding over the top of the wing first speeds up and then slows down again.

Rate of change 'following the fluid'

This notion is of fundamental importance in fluid dynamics.

Let $f(x, y, z, t)$ denote some quantity of interest in the fluid motion. It could, for example, be one component of the velocity $\boldsymbol{u}$, or it could be the density ρ. Note first that $\partial f/\partial t$ means the rate of change of f at fixed x, y, and z, i.e. at a fixed position in space.

In contrast, the rate of change of f 'following the fluid', which we denote by $\mathrm{D}f/\mathrm{D}t$, is

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\mathrm{d}}{\mathrm{d}t} f[x(t), y(t), z(t), t],$$

where $x(t)$, $y(t)$, and $z(t)$ are understood to change with time at the local flow velocity $\boldsymbol{u}$:

$$\mathrm{d}x/\mathrm{d}t = u, \qquad \mathrm{d}y/\mathrm{d}t = v, \qquad \mathrm{d}z/\mathrm{d}t = w,$$

so as to 'follow the fluid'. A simple application of the chain rule gives

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial f}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial f}{\partial t},$$

whence

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z},$$

i.e.

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla)f. \tag{1.6}$$

By applying eqn (1.6) to the velocity components u, v, and w in turn it follows, in particular, that the *acceleration* of the fluid element at $\boldsymbol{x}$ is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}. \tag{1.7}$$

As an immediate check on eqn (1.7) consider fluid in uniform rotation with angular velocity Ω, so that

$$u=-\Omega y, \qquad v=\Omega x, \qquad w=0.$$

Now $\partial \boldsymbol{u}/\partial t$ is zero, because the flow is steady, but

$$\begin{aligned}(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}&=\left(-\Omega y\frac{\partial}{\partial x}+\Omega x\frac{\partial}{\partial y}\right)(-\Omega y,\, \Omega x,\, 0)\\ &=-\Omega^2(x,\, y,\, 0).\end{aligned}$$

This is just as expected; it represents the familiar centrifugal acceleration $\Omega^2 r$ towards the rotation axis.

According to eqn (1.6) in any *steady* flow the rate of change of f following a fluid element is $(\boldsymbol{u}\cdot\nabla)f$, and it is quite easy to see why this should be so. Let $\boldsymbol{e}_s$ denote a unit vector which is always parallel to the streamlines and in the same sense as the flow. Then

$$\boldsymbol{u}\cdot\nabla f=|\boldsymbol{u}|\,\boldsymbol{e}_s\cdot\nabla f=|\boldsymbol{u}|\,\frac{\partial f}{\partial s},$$

where s denotes distance along a streamline. Now, $\partial f/\partial s$ is the rate of change of f with distance along a streamline, so multiplying it by the flow speed $|\boldsymbol{u}|$ evidently gives the rate of change with time as we follow a fluid element along that streamline.

The equation

$$(\boldsymbol{u}\cdot\nabla)f=0, \tag{1.8}$$

which arises at some important stages in the following theory, thus implies that f is constant *along a streamline*. It should be emphasized that eqn (1.8) offers no information at all about whether f might be a different constant on different streamlines. Suppose, for instance, that the flow is everywhere in the x-direction, so that eqn (1.8) reduces to $\partial f/\partial x=0$. This equation says that f is independent of x, but it contains no implication about how f might depend on y, z, or t.

Likewise, the equation

$$\frac{Df}{Dt} = 0, \tag{1.9}$$

which also arises in the following theory, implies that f is a constant *for a particular fluid element*, and this follows directly from the definition of Df/Dt above. It does not preclude different elements having different values of f; it just implies that each such element will retain whatever value of f it started with.

Finally, it is worth remarking that there will be occasions on which we wish to follow not just an infinitesimal fluid element but a finite blob consisting always of the same fluid particles. Such a blob, which will of course deform as it moves about, is typically called a 'material' volume in the literature, but we shall freely describe it instead as 'dyed', with the understanding, of course, that no diffusion of this imaginary dye is envisaged. Such terminology can become rather colourful, but if it evokes a sharp mental picture of a moving and deforming blob of fluid, as opposed to some region fixed in space, it serves its purpose.

1.3. Equations of motion for an ideal fluid

In this text we define an *ideal fluid* as one with the following properties:

(i) It is *incompressible*, so that no 'dyed' blob of fluid can change in volume as it moves.

(ii) The density ρ (i.e. the mass per unit volume) is a constant, the same for all fluid elements and for all time t.

(iii) The force exerted across a geometrical surface element $\boldsymbol{n}\,\delta S$ within the fluid is

$$p\boldsymbol{n}\,\delta S, \tag{1.10}$$

where $p(x, y, z, t)$ is a scalar function, independent of the normal $\boldsymbol{n}$, called the *pressure*. (To be more precise, eqn (1.10) is the force exerted *on* the fluid into which $\boldsymbol{n}$ is pointing *by* the fluid on the other side of δS.)

There is, of course, no such thing in practice as an ideal fluid. All fluids are to some extent compressible, and all fluids are to some extent *viscous*, so that adjacent fluid elements exert both normal and tangential forces on one another across their common interface. For the time being, however, we explore some consequences of the assumptions (i)–(iii).

To examine the implications of (i), consider a fixed closed surface S drawn in the fluid, with unit outward normal $\boldsymbol{n}$. Fluid will be entering the enclosed region V at some places on S, and leaving it at others. The velocity component along the outward normal is $\boldsymbol{u} \cdot \boldsymbol{n}$, so the volume of fluid leaving through a small surface element δS in unit time is $\boldsymbol{u} \cdot \boldsymbol{n}\, \delta S$. The net volume rate at which fluid is leaving V is therefore

$$\int_S \boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}S.$$

But this must plainly be zero for an incompressible fluid, and on using the divergence theorem we find that

$$\int_V \nabla \cdot \boldsymbol{u}\, \mathrm{d}V = 0.$$

Now, this must be true for all regions V within the fluid. Suppose, then, that $\nabla \cdot \boldsymbol{u}$ is greater than zero at some point in the fluid. Assuming that it is continuous, $\nabla \cdot \boldsymbol{u}$ will be greater than zero in some small sphere around that point, and by taking V to be such a sphere we violate the above equation. The same applies if $\nabla \cdot \boldsymbol{u}$ is negative at some point. We thus conclude that

$$\nabla \cdot \boldsymbol{u} = 0 \tag{1.11}$$

everywhere in the fluid.

This *incompressibility condition* is an important constraint on the velocity field $\boldsymbol{u}$ in virtually the whole of this book.†

To examine the implications of (iii) consider a surface S enclosing a 'dyed' blob of fluid. The force exerted by the surrounding fluid across any surface element δS is, by hypothesis, given by eqn (1.10), so that the net force exerted on the dyed blob is

$$-\int_S p\boldsymbol{n}\, \mathrm{d}S = -\int_V \nabla p\, \mathrm{d}V,$$

† Air is, of course, highly compressible, but it can behave like an incompressible fluid if the flow speed is much smaller than the speed of sound (see p. 58).

where we have used the identity (A.14)—see the Appendix (the negative sign arises because $\boldsymbol{n}$ points out of S). Now, provided that ∇p is continuous it will be almost constant over a *small* blob of fluid of volume δV. The net force on such a small blob due to the pressure of the surrounding fluid will therefore be $-\nabla p\, \delta V$.

Euler's equations of motion

We are now in a position to apply the principle of linear momentum to a small 'dyed' blob of fluid of volume δV. Allowing for the presence of a gravitational body force per unit mass $\boldsymbol{g}$, the total force on the blob is

$$(-\nabla p + \rho \boldsymbol{g})\, \delta V.$$

This force must be equal to the product of the blob's mass (which is conserved) and its acceleration, i.e. to

$$\rho\, \delta V \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}.$$

We thus obtain

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \boldsymbol{g}, \qquad (1.12)$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

as the basic equations of motion for an ideal fluid. They are known as *Euler's equations*, and written out in full they become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

i.e. four scalar equations for four unknowns: u, v, w, and p. In dealing with the gravitational term we have momentarily taken the z-axis vertically upward, setting $\boldsymbol{g} = (0, 0, -g)$.

Now, the gravitational force, being conservative, can be written as the gradient of a potential:

$$\boldsymbol{g} = -\nabla\chi. \tag{1.13}$$

(In the above case, $\chi = gz$.) Using the expression (1.7) for the fluid acceleration we may rewrite eqn (1.12) in the form†

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \chi\right),$$

where we have used the assumption that ρ is constant. Furthermore, it can be helpful to use the identity

$$(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} + \nabla(\tfrac{1}{2}\boldsymbol{u}^2)$$

to cast the momentum equation into the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi\right). \tag{1.14}$$

The Bernoulli streamline theorem

If the flow is steady, eqn (1.14) reduces to

$$(\nabla \wedge \boldsymbol{u}) \wedge \boldsymbol{u} = -\nabla H,$$

where

$$H = \frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi. \tag{1.15}$$

On taking the dot product with $\boldsymbol{u}$ we obtain

$$(\boldsymbol{u} \cdot \nabla)H = 0, \tag{1.16}$$

† The way in which $p/\rho + \chi$ appears as a combination is significant; there will be many circumstances in this book in which gravity simply modifies the pressure distribution in the fluid and does nothing to change the velocity $\boldsymbol{u}$. Thus when we speak occasionally of 'ignoring' gravity, or of gravitational body forces being 'absent', what we often mean is that separate allowance may be made for gravity simply by subtracting $\rho\chi$ from the pressure field. This is emphatically not the case, however, if there is a free surface—as with water waves in Chapter 3—or if ρ is not constant—as in §3.8 and §9.3.

so

If an ideal fluid is in steady flow,
then H is constant along a streamline.

In the absence of gravity it follows that $p + \frac{1}{2}\rho \boldsymbol{u}^2$ is constant along a streamline in steady flow.

The above theorem says nothing about H being the same constant on different streamlines, only that it remains constant along each one. There is, however, one important circumstance in which H is constant throughout the whole flow field, and this now follows.

DEFINITION. An *irrotational* flow is one for which

$$\nabla \wedge \boldsymbol{u} = 0. \tag{1.17}$$

The Bernoulli theorem for irrotational flow

If the flow is steady and irrotational, then eqn (1.14) reduces to $\nabla H = 0$, so H is independent of x, y, and z, as well as t. Thus

If an ideal fluid is in steady irrotational flow,
then H is constant throughout the whole flow field.

Whether this result is of any value rests, evidently, on whether irrotational flows are of any real interest in practice. We address this matter in the next section.

1.4. Vorticity: irrotational flow

The *vorticity* $\boldsymbol{\omega}$ is defined as

$$\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}, \tag{1.18}$$

and it is a concept of central importance in fluid dynamics. The vorticity is, by definition, zero for an *irrotational* flow.

We consider vorticity first in the context of two-dimensional flow, for if

$$\boldsymbol{u} = [u(x, y, t), v(x, y, t), 0]$$

then $\boldsymbol{\omega}$ is $(0, 0, \omega)$, where

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{1.19}$$

Interpretation of vorticity in 2-D flow

Consider two short fluid line elements AB and AC *which are perpendicular at a certain instant,* as in Fig. 1.3. Note that the y-component of velocity at B exceeds that at A by

$$v(x+\delta x, y, t) - v(x, y, t) \doteqdot \frac{\partial v}{\partial x}\,\delta x,$$

so that $\partial v/\partial x$ represents the instantaneous angular velocity of the fluid line element AB. Likewise, $\partial u/\partial y$ represents the instantaneous angular velocity (in the opposite sense) of the line element AC. Thus at any point of the flow field

$$\tfrac{1}{2}\omega = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

represents the average angular velocity of two short fluid line elements that happen, at that instant, to be mutually perpendicular. In this precise sense the vorticity ω acts as a measure of the *local* rotation, or spin, of fluid elements.

We emphasize that vorticity has nothing directly to do with any global rotation of the fluid. Take, for example, the shear flow of

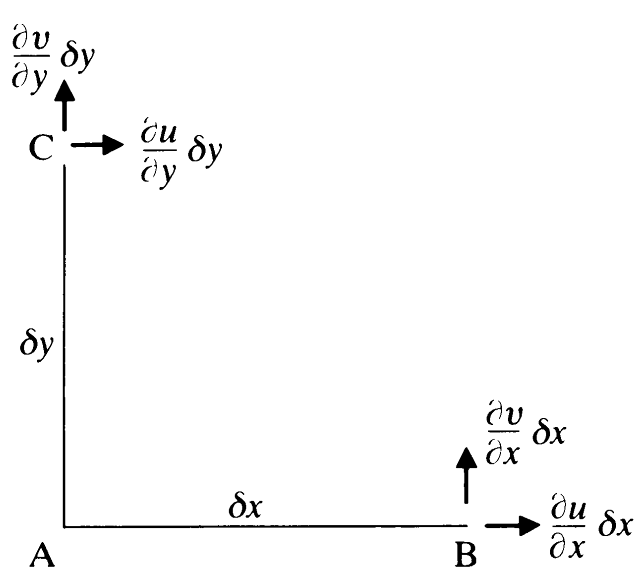


Fig. 1.3. Sketch for the interpretation of vorticity in 2-D flow. The velocity components shown are relative to the fluid particle at A.

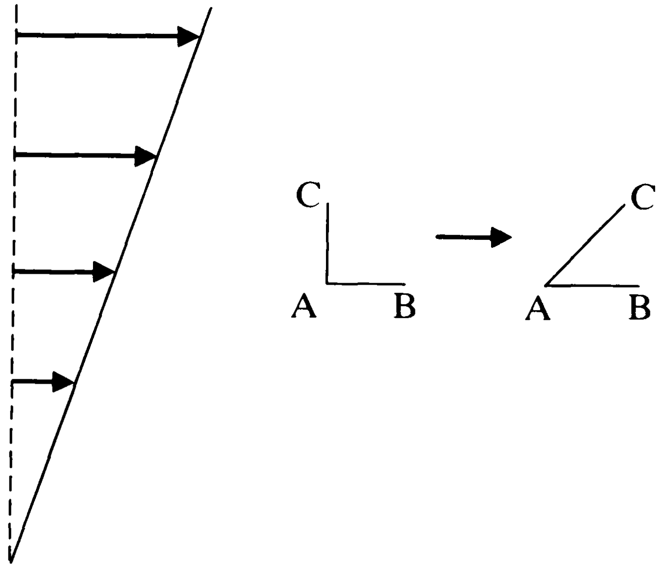


Fig. 1.4. Deformation of two short, momentarily perpendicular fluid line elements in a shear flow.

Fig. 1.4, in which

$$\boldsymbol{u} = (\beta y, 0, 0), \tag{1.20}$$

where β is a constant. The fluid is certainly not rotating globally in any sense, but it has vorticity:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\beta;$$

and two momentarily perpendicular line elements, AB and AC, orientated as shown plainly have an average angular velocity (in fact, of $-\frac{1}{2}\beta$), because while that of AB is zero that of AC is not.

A more colourful example of the distinction between vorticity and global rotation of the fluid is provided by the so-called *line vortex* flow given in cylindrical polar coordinates (r, θ, z) by

$$\boldsymbol{u} = \frac{k}{r}\boldsymbol{e}_\theta, \tag{1.21}$$

where k is a constant. To find the vorticity of this flow we need the expression (A.32) for $\nabla \wedge \boldsymbol{u}$ in cylindrical polar coordinates:

$$\nabla \wedge \boldsymbol{u} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ u_r & ru_\theta & u_z \end{vmatrix}.$$

Plainly, then, the vorticity is zero except at $r = 0$, where neither $\boldsymbol{u}$ nor $\nabla \wedge \boldsymbol{u}$ is defined. Thus although the fluid is clearly rotating in a global sense the flow is in fact *irrotational*, since $\nabla \wedge \boldsymbol{u} = 0$, except on the axis. This is quite understandable if we consider two momentarily perpendicular fluid line elements, AB and AC, at $\theta = 0$ in Fig. 1.5. Clearly AC is rotating in an anticlockwise sense, because it will continue to lie along the circular streamline as time proceeds, but AB is rotating clockwise because of the decrease of u_θ with r in eqn (1.21). This particular fall-off of u_θ with r is, apparently, just the correct one—neither too slow nor too rapid—to ensure that AB has an equal and opposite angular velocity to AC at the instant they are perpendicular, so that their average angular velocity is zero.

We keep emphasizing the instantaneous nature of this conclusion about zero average angular velocity because two fluid line elements such as AB and AC in Fig. 1.5 will not remain perpendicular as they get carried about by the flow, and as soon as this happens we have no cause to conclude from the irrotationality of the flow that their average angular velocity should any longer be zero.

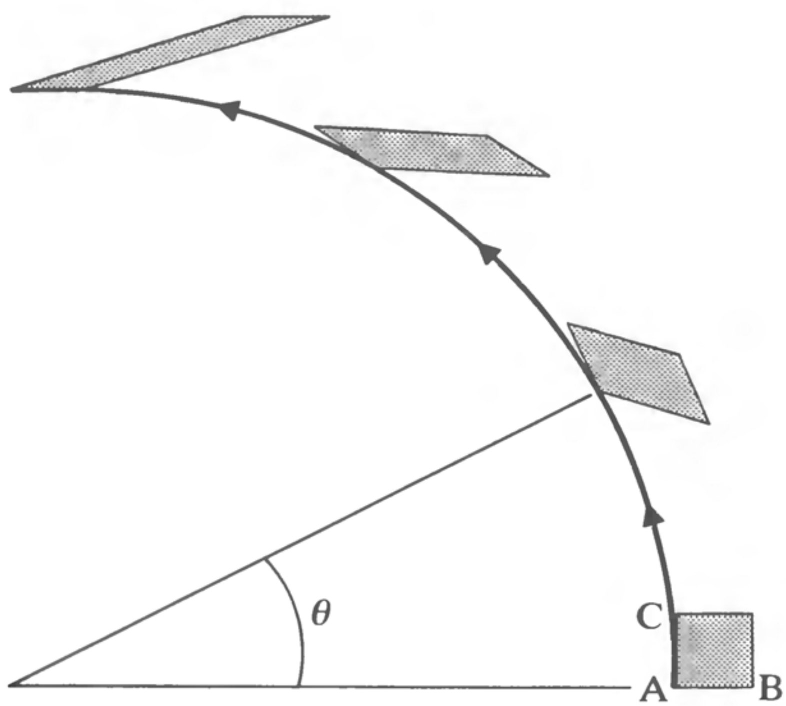
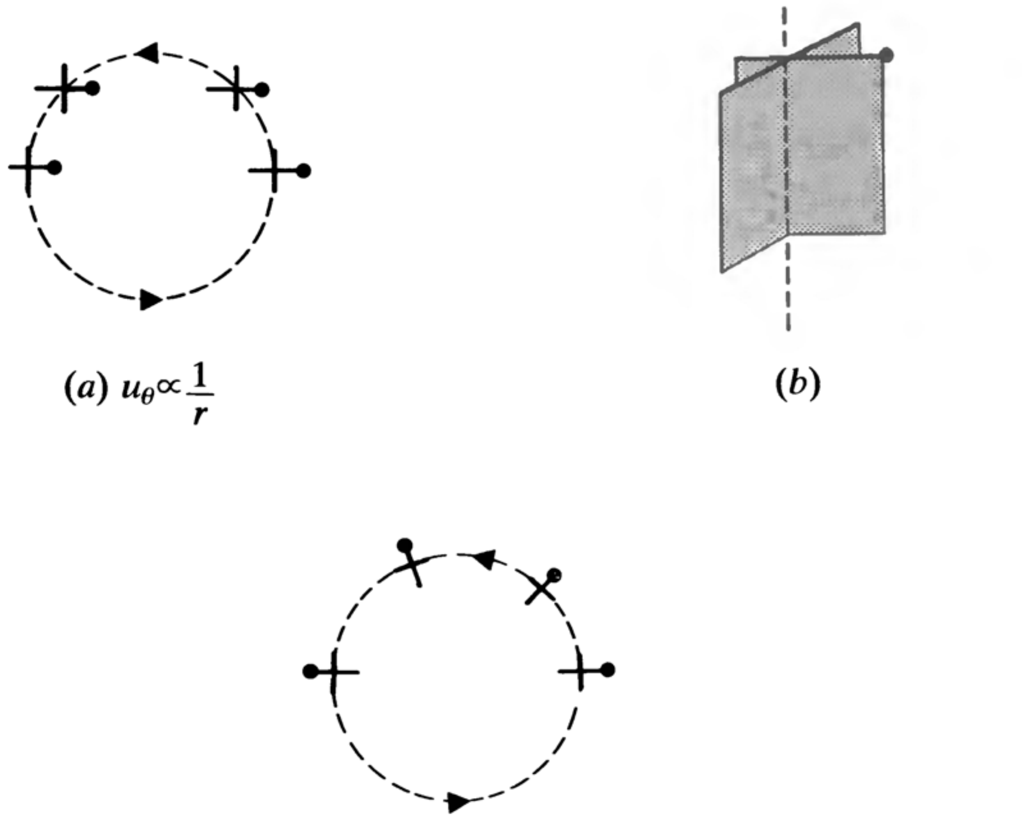


Fig. 1.5. The fate of a small square fluid element in a line vortex flow. The size of the element has been greatly exaggerated for the sake of clarity; an unfortunate consequence is that B does not look as if it is following a circular path.



(*a*) $u_\theta \propto \frac{1}{r}$ (*b*)

(*c*) $u_\theta \propto r$

Fig. 1.6. A crude 'vorticity meter' (*b*), and its behaviour when immersed in a line vortex flow (*a*) and a uniformly rotating flow (*c*).

What we have sketched in Fig. 1.6(*a*), then, is not what happens to two momentarily dyed fluid elements, AB and AC, as they get swept round but what would happen if we were to immerse in the fluid a small 'vorticity meter' consisting of two short, rigid vanes fixed at right angles to each other, as in Fig. 1.6(*b*). We have marked one tip of one of the vanes, and in Fig. 1.6(*a*) we see that this device would not rotate in this particular (line vortex) flow, even though its axis would of course get swept round on a circular streamline. This behaviour may be seen in the bath by observing closely the strong vortex that may occur as the water goes down the plug-hole. The azimuthal velocity u_θ varies roughly as r^{-1} over a fair distance from the axis, and a crude but simple vorticity meter which serves the purpose consists of a pair of short wooden line elements shaved off a matchstick, sellotaped together at right angles and floated on the surface.

However, if such a vorticity meter were to be inserted in the flow

$$\boldsymbol{u} = \Omega r \boldsymbol{e}_\theta, \tag{1.22}$$

Ω being a constant, the result would of course be as in Fig. 1.6(*c*), because the device would get carried around just as if it were embedded in a rigid body. Its angular velocity would evidently be Ω, the same as the uniform angular velocity of the fluid as a whole, and the vorticity of the flow is therefore $(0, 0, 2\Omega)$, as may be confirmed by direct calculation of $\nabla \wedge \boldsymbol{u}$.

By putting the two flows in Fig. 1.6 together in the following way:

$$u_\theta = \begin{cases} \Omega r, & r < a, \\ \dfrac{\Omega a^2}{r}, & r > a, \end{cases}$$

$$u_r = u_z = 0, \tag{1.23}$$

we obtain a so-called 'Rankine vortex', which serves as a simple model for a real vortex such as that in Fig. 1.1. Real vortices are typically characterized by fairly small vortex 'cores' in which, by definition, the vorticity is concentrated, while outside the core the flow is essentially irrotational. The core is not usually exactly circular, of course; nor is the vorticity usually uniform within it. In these two respects the Rankine vortex of Fig. 1.7 is only an idealized model.

We have now said a fair amount about vorticity, albeit strictly

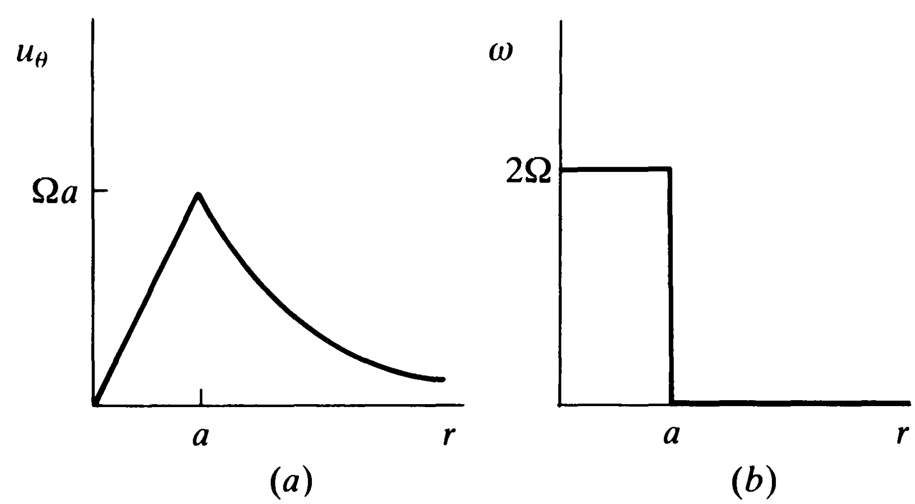


Fig. 1.7. Distribution of (*a*) azimuthal velocity u_θ and (*b*) vorticity ω in a Rankine vortex.

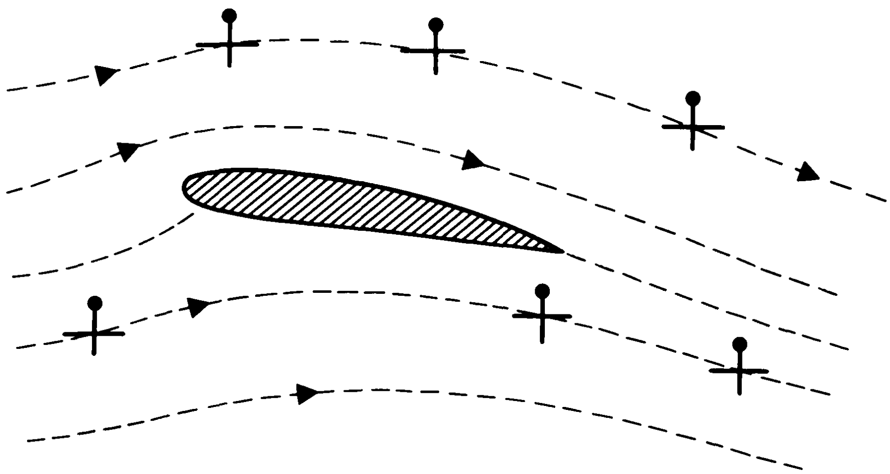


Fig. 1.8. The behaviour of a small 'vorticity meter' placed in the steady flow past a fixed wing at small angle of attack. The flow is clearly irrotational.

in the context of two-dimensional flow. We have discussed in particular detail the absence of vorticity, i.e. irrotational flow. At this stage, before the development seems to be getting rather a long way from our starting point (the experiment in §1.1), we should say that steady flow past a wing at small angles of incidence α *is* typically irrotational, as indicated in Fig. 1.8.

Why this should be so emerges from the Euler equations in a very elegant manner, as we now see.

1.5. The vorticity equation

In its form (1.14), Euler's equation may be written

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \wedge \boldsymbol{u} = -\nabla H,$$

and on taking the curl we obtain

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \nabla \wedge (\boldsymbol{\omega} \wedge \boldsymbol{u}) = 0. \tag{1.24}$$

Using the vector identity (A.6) this becomes

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} + \boldsymbol{\omega}\nabla \cdot \boldsymbol{u} - \boldsymbol{u}\nabla \cdot \boldsymbol{\omega} = 0.$$

Now the fourth term vanishes because the fluid is incompressible, while the fifth term vanishes because div curl = 0. We therefore have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u},$$

or, alternatively,

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}. \tag{1.25}$$

This *vorticity equation* is extremely valuable. Note that the pressure has been eliminated; eqn (1.25) involves only $\boldsymbol{u}$ and $\boldsymbol{\omega}$, which are, of course, related by

$$\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u}.$$

In particular, if the flow is *two-dimensional,* so that

$$\boldsymbol{u} = [u(x, y, t), v(x, y, t), 0] \tag{1.26}$$

and

$$\boldsymbol{\omega} = (0, 0, \omega),$$

then

$$(\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = \omega \frac{\partial \boldsymbol{u}}{\partial z} = 0.$$

It then follows that

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = 0, \tag{1.27}$$

and we thus conclude, referring back to eqn (1.9), that

In the two-dimensional flow of an ideal fluid subject to a conservative body force **g** *the vorticity* ω *of each individual fluid element is conserved.* (1.28)

This result has important applications, which we discuss in Chapter 5. In the particular case of steady flow, eqn (1.27) reduces to

$$(\boldsymbol{u} \cdot \nabla)\omega = 0 \tag{1.29}$$

and consequently

In the steady, two-dimensional flow of an ideal fluid subject to a conservative body force **g** *the vorticity* ω *is constant along a streamline.* (1.30)

This, then, is the reason why the steady flow in Fig. 1.8 is irrotational. Note first that there are no regions of closed streamlines in the flow; all the streamlines can be traced back to $x = -\infty$. Now, the vorticity is constant along each one, and hence equal on each one to whatever it is on that particular streamline at $x = -\infty$. As the flow is uniform at $x = -\infty$ the vorticity is zero on all streamlines there. Hence it is zero throughout the flow field in Fig. 1.8.

1.6. Steady flow past a fixed wing

In Fig. 1.9 we show typical measured pressure distributions on the upper and lower surfaces of a fixed wing in steady flow. The pressures on the upper surface are substantially lower than the free-stream value p_∞, while those on the lower surface are a little higher than p_∞. In fact, then, the wing gets most of its lift from a suction effect on its upper surface.

But why is it that the pressures above the wing are less than those below? Well, because the flow is irrotational, the Bernoulli theorem tells us that $p + \frac{1}{2}\rho \boldsymbol{u}^2$ is constant throughout the flow. Explaining the pressure differences, and hence the lift on the

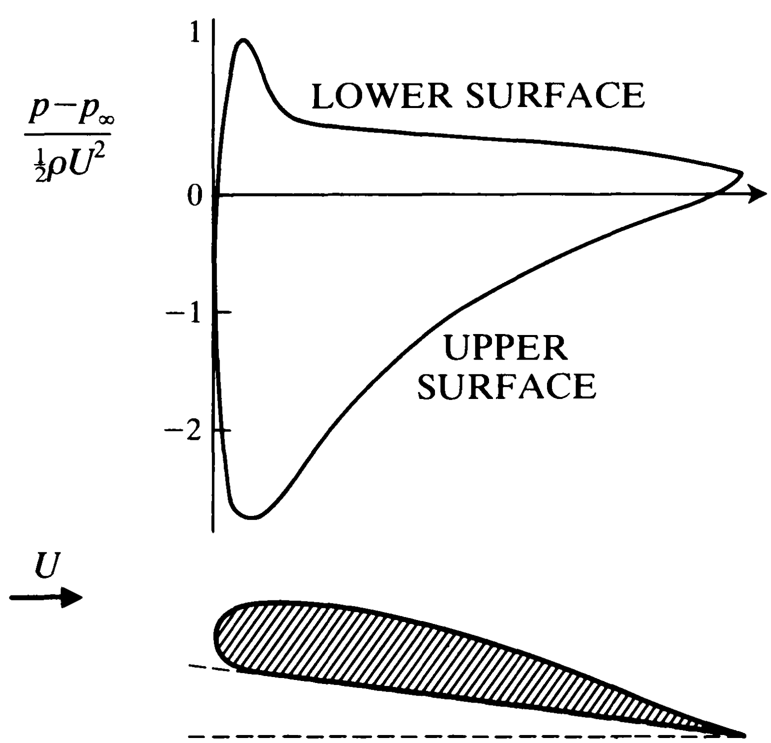


Fig. 1.9. Typical pressure distribution on a wing in steady flow.

wing, thus reduces to explaining why (as in Fig. 1.2) the flow speeds above the wing are greater than those below.

Let us first dispose of one bogus explanation that occasionally appears, namely that the air on the top of the wing flows faster 'because it has farther to go'. There are many woolly aspects to this argument, but it seems to turn principally on the notion that two neighbouring fluid elements, after parting to go their separate ways round the wing, meet up again at the trailing edge, and this is demonstrably false (see Fig. 2.4).

The right way forward to an explanation of the higher flow speeds above the wing is in terms of the concept of circulation.

Circulation

Let C be some closed curve lying in the fluid region. Then the circulation Γ round C is defined as

$$\Gamma = \int_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}. \tag{1.31}$$

At first sight, perhaps, there cannot be any circulation in an irrotational flow, for Stokes's theorem gives

$$\int_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = \int_S (\nabla \wedge \boldsymbol{u}) \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{1.32}$$

and an irrotational flow is, by definition, one for which $\nabla \wedge \boldsymbol{u}$ is zero. But such an argument holds only if the closed curve C in question can be spanned by a surface S which lies wholly in the region of irrotational flow. Thus in the two-dimensional context of Fig. 1.8, for example, for which eqn (1.32) reduces to

$$\Gamma = \int_C u \, \mathrm{d}x + v \, \mathrm{d}y = \int_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y, \tag{1.33}$$

it is true that Γ must be zero for any closed curve C not enclosing the wing, but the argument fails for any closed curve that does enclose the wing. The most that can be said about such circuits is that they all have the same value of Γ (Exercise 1.6).

Circulation round a wing is permissible, then, in a steady irrotational flow; but the question still arises as to why there should be any, and, in particular, why it should be negative, corresponding to larger flow speeds above the wing than below.

The Kutta–Joukowski hypothesis

In the case of a wing with a sharp trailing edge, one good reason for non-zero circulation Γ is that there would otherwise be a singularity in the velocity field. The irrotational flow past a wing with $\Gamma = 0$ is sketched in Fig. 1.10(*a*), but the velocity is infinite at the trailing edge where, loosely speaking, the fluid is having a hard time turning the corner. We show in Chapter 4 that only for one value of the circulation, Γ_K say, is the flow speed finite at the trailing edge, as in Fig. 1.10(*b*). It is natural to hope that this particular irrotational flow will correspond to the steady flow that is actually observed; this is the *Kutta–Joukowski hypothesis.*

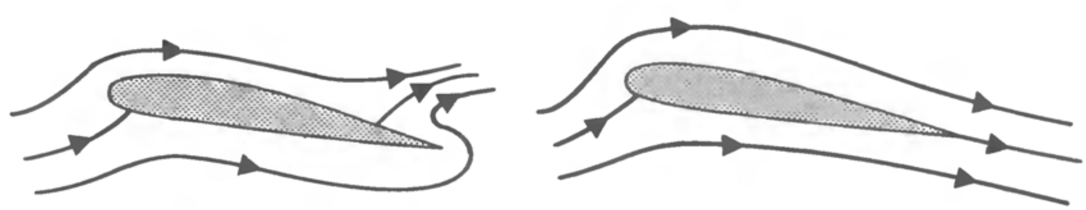
This hypothesis is inevitably somewhat *ad hoc*, resting as it does on the unsatisfactory state of affairs that would otherwise arise because of the sharp trailing edge. (How are we to decide between all the different irrotational flows if the trailing edge is not sharp?) It is, nonetheless, one of the key steps in the development of aerodynamics, and gives results which are in excellent accord with experiment, as we shall shortly see.

The critical value Γ_K depends, naturally, on the flow speed at infinity U and on the size, shape, and orientation of the wing. In Chapter 4 we show that if the wing is thin and symmetrical, of length L, making an angle α with the oncoming stream, then

$$\Gamma_K \doteqdot -\pi U L \sin \alpha. \tag{1.34}$$

Lift

According to ideal flow theory, the *drag* on the wing (the force parallel to the oncoming stream) is zero, but the *lift* (the force



(*a*) (*b*)

Fig. 1.10. Irrotational flow past a fixed wing with (*a*) $\Gamma = 0$ and (*b*) $\Gamma = \Gamma_K < 0$.

perpendicular to the stream) is

$$\mathscr{L} = -\rho U \Gamma. \tag{1.35}$$

This *Kutta–Joukowski Lift Theorem* is proved in §4.11.

That negative Γ should give positive lift is entirely natural; we have argued as much in the preceding sections. As a precise theorem, however, eqn (1.35) is rather extraordinary, as it holds for irrotational flow (uniform at infinity) past a two-dimensional body of any size or shape; $\mathscr{L}$ depends on the size and shape of the body only inasmuch as Γ does. For the thin symmetrical wing of Fig. 1.10(*b*), for example, with Γ as in eqn (1.34) by the Kutta–Joukowski condition, the lift is

$$\mathscr{L} \doteqdot \pi \rho U^2 L \sin \alpha. \tag{1.36}$$

Agreement with experiment is good provided that α is only a few degrees (Fig. 1.11). Thereafter the measured lift falls dramatically and diverges substantially from the predictions of inviscid theory, for reasons to be discussed later. The angle α at which this divergence begins may be anywhere between about 6° and 12°, depending on the shape of the wing (see, e.g., Nakayama 1988, pp. 76–80).

Accounting for the flow past a wing at small angles of attack α is nevertheless one of the great, and practically important, successes of ideal-flow theory.

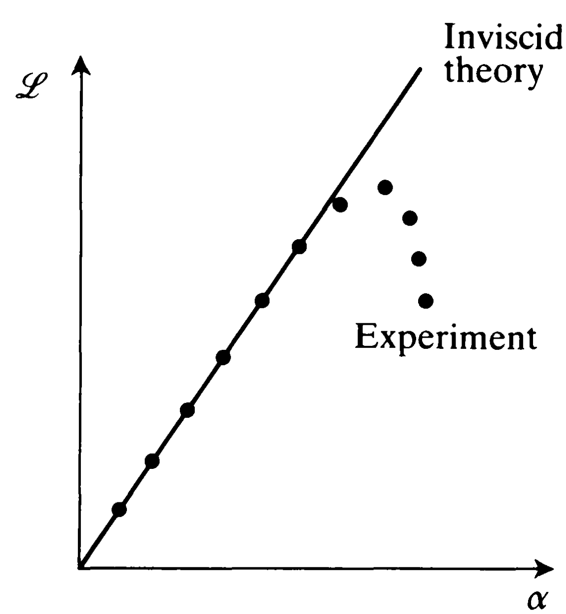


Fig. 1.11. Lift on a symmetric aerofoil.

1.7. Concluding remarks

In this chapter we have introduced some of the basic concepts of fluid dynamics and, at the same time, given some indication of how they figure in one particular branch of the subject, namely aerodynamics. Our treatment of this branch has inevitably been sketchy.

We have, for instance, focused wholly on 2-D aerodynamics, yet any real wing, no matter how long, has ends, and important new phenomena then arise. The circulation round a circuit such as C in Fig. 1.12(a) is essentially that predicted by the 2-D theory (i.e. eqn (1.34)), but plainly the flow cannot be everywhere irrotational, because C can now be spanned by a surface S which lies wholly in the fluid. Indeed, from Stokes's theorem (1.32) we deduce that there must be a positive flux of vorticity out of S, and this is in practice observed as a concentrated *trailing vortex* emanating from the wing-tip as shown. The higher the lift (and

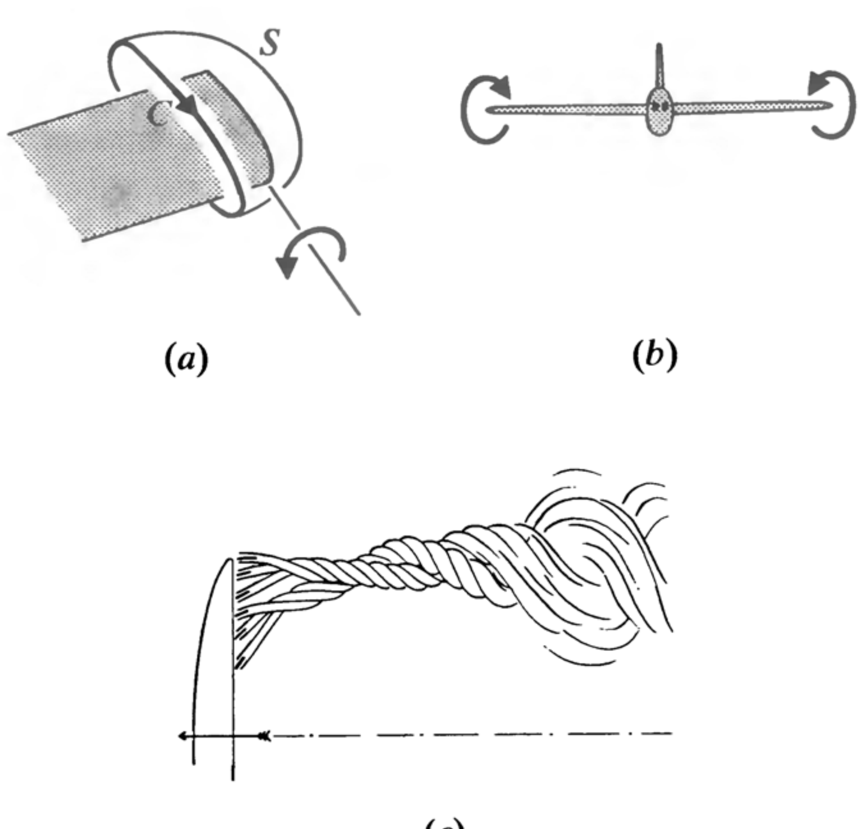


Fig. 1.12. Trailing vortices: (a) definition sketch for application of Stokes's theorem; (b) view from some distance ahead of the aircraft; (c) the original drawing from Lanchester's *Aerodynamics* (1907).

therefore the circulation), the stronger the trailing vortices. Furthermore, the presence of these trailing vortices results in a drag on the wing, even on ideal flow theory, for as they lengthen they contain more and more kinetic energy, and creating all this kinetic energy takes work.

But even within a purely two-dimensional framework we have left some key questions unanswered. We indicated how the Kutta–Joukowski hypothesis provides a rational, although *ad hoc,* basis for deciding the circulation round an aerofoil in steady flight, and we have noted that this gives good agreement with experiment. Yet we have given no account of the dynamical processes by which that circulation is *generated* when the aerofoil starts from a state of rest. It arises, in fact, in response to the 'starting vortex' in §1.1, but why this should be so is far from obvious, and rests on one of the deepest theorems in the subject (§5.1).

Again, the sceptical reader may even be asking: 'But what is all this business about a starting vortex? If the aerofoil and fluid are initially at rest, the vorticity ω is initially zero for each fluid element. By eqn (1.27) it remains zero for each fluid element, even when the aerofoil has been started into motion. Therefore there should not be a starting vortex.'

This is a legitimate conclusion—on the basis of ideal flow theory. While that theory accounts well for the steady flow past an aerofoil, the explanation of how that flow became established involves *viscous* effects in a crucial way.

If this provokes the response: 'But air isn't very viscous, is it?', the answer is, 'No, in some sense air is hardly viscous at all'. Yet, as we shall see, viscous effects are sufficiently subtle that the shedding of the vortex in §1.1, while being an essentially viscous process, would occur no matter how small the viscosity of the fluid happened to be.

Exercises

1.1. Whether a fluid is incompressible or not, each element must conserve its mass as it moves. Consider the rate of mass flow through a fixed closed surface S drawn in the fluid, and use an argument similar to that on p. 7 to show that this *conservation of mass* implies

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{1.37}$$

where $\rho(\boldsymbol{x}, t)$ denotes the (variable) density of the fluid. Show too that this equation may alternatively be written

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla \cdot \boldsymbol{u} = 0. \tag{1.38}$$

It follows that if $\nabla \cdot \boldsymbol{u} = 0$, then $\mathrm{D}\rho/\mathrm{D}t = 0$. What does this mean, exactly, and does it make sense?

1.2. An ideal fluid is rotating under gravity g with constant angular velocity Ω, so that relative to fixed Cartesian axes $\boldsymbol{u} = (-\Omega y, \Omega x, 0)$. We wish to find the surfaces of constant pressure, and hence the surface of a uniformly rotating bucket of water (which will be at atmospheric pressure).

'By Bernoulli,' $p/\rho + \frac{1}{2}\boldsymbol{u}^2 + gz$ is constant, so the constant pressure surfaces are

$$z = \text{constant} - \frac{\Omega^2}{2g}(x^2 + y^2).$$

But this means that the surface of a rotating bucket of water is at its highest in the middle. What is wrong?

Write down the Euler equations in component form, integrate them directly to find the pressure p, and hence obtain the correct shape for the free surface.

1.3. Find the pressure p both inside and outside the core of the Rankine vortex (1.23). Show that the pressure at $r = 0$ is lower than that at $r = \infty$ by an amount $\rho\Omega^2 a^2$ (hence the very low pressure in the centre of a tornado). Deduce that if there is a free surface to the fluid and gravity is acting, then the surface at $r = 0$ is a depth $\Omega^2 a^2/g$ below the surface at $r = \infty$ (hence the dimples in a cup of tea accompanying the vortices that are shed by the edges of the spoon).

1.4. Take the Euler equation for an incompressible fluid of constant density, cast it into an appropriate form, and perform suitable operations on it to obtain the *energy equation*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \tfrac{1}{2}\rho\boldsymbol{u}^2 \,\mathrm{d}V = -\int_S (p' + \tfrac{1}{2}\rho\boldsymbol{u}^2)\boldsymbol{u} \cdot \boldsymbol{n} \,\mathrm{d}S,$$

where V is the region enclosed by a fixed closed surface S drawn in the fluid, and p' denotes $p + \rho\chi$, the non-hydrostatic part of the pressure field.

1.5. For an inviscid fluid we have Euler's equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega} \wedge \boldsymbol{u} + \nabla(\tfrac{1}{2}\boldsymbol{u}^2) = -\frac{1}{\rho}\nabla p - \nabla\chi,$$

and, whether or not the fluid is incompressible, we also have conservation of mass (Exercise 1.1):

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot \boldsymbol{u} = 0.$$

Show that

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}}{\rho} \cdot \nabla\right)\boldsymbol{u} - \frac{1}{\rho}\nabla\left(\frac{1}{\rho}\right) \wedge \nabla p. \tag{1.39}$$

Deduce that, if p is a function of ρ alone, the vorticity equation is exactly as in the incompressible, constant density case, except that $\boldsymbol{\omega}$ is replaced by $\boldsymbol{\omega}/\rho$.

1.6. Show that the circulation is the same round all simple closed circuits enclosing the wing in Fig. 1.8. (Hint: sketch two such circuits, and then make a construction so as to create a single closed circuit that does not enclose the wing.)

1.7. Sketch the streamlines for the flow

$$u = \alpha x, \qquad v = -\alpha y, \qquad w = 0,$$

where α is a positive constant.

Let the concentration of some pollutant in the fluid be

$$c(x, y, t) = \beta x^2 y \mathrm{e}^{-\alpha t},$$

for $y > 0$, where β is a constant. Does the pollutant concentration for any particular fluid element change with time?

An alternative way of describing any flow is to specify the position $\boldsymbol{x}$ of each fluid element at time t in terms of the position $\boldsymbol{X}$ of that element at time $t = 0$. For the above flow this 'Lagrangian' description is

$$x = X\mathrm{e}^{\alpha t}, \qquad y = Y\mathrm{e}^{-\alpha t}, \qquad z = Z.$$

Verify by direct calculation that

$$\left(\frac{\partial \boldsymbol{x}}{\partial t}\right)_{\boldsymbol{X}} = \boldsymbol{u}, \qquad \left(\frac{\partial \boldsymbol{u}}{\partial t}\right)_{\boldsymbol{X}} = \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}$$

in this particular case. Why are these results true in general?

Write c as a function of X, Y, and t.

1.8. Consider the unsteady flow

$$u = u_0, \qquad v = kt, \qquad w = 0,$$

where u_0 and k are positive constants. Show that the streamlines are straight lines, and sketch them at two different times. Also show that any fluid particle follows a parabolic path as time proceeds.