On the Stability, or Instability, of certain Fluid Motions. By

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In a former communication to the Society on the "Instability of Jets,"* I applied a method due to Sir W. Thomson, to calculate the manner of falling away from equilibrium of jets bounded by one or more surfaces of discontinuity. Such interest as these investigations possessed was due principally to the possibility of applying their results to the explanation of certain acoustical phenomena relating to sensitive flames and smoke jets. But it soon appeared that in one important respect the calculations failed to correspond with the facts.

To fix the ideas, let us take the case of an originally plane surface of separation, on the two sides of which the fluid moves with equal and opposite constant velocities ($\pm V$). In equilibrium, the elevation h, at every point x along the surface, is zero. It is proved that, if initially the surface be a rest in the form defined by $h = H \cos \kappa x$, then, after a time t, its form is given by

$$h = H \cos \kappa x \cosh \kappa Vt \quad \text{..........................(1)},$$

provided that, throughout the whole time contemplated, the disturbance is small. In the same sense as that in which the frequency of vibration measures the stability of a system vibrating about a configuration of stable equilibrium, so the coefficient κV of t, in equations such as (1), measures the instability of an unstable system; and we see, in the present case, that the instability increases without limit with κ; that is to say, the shorter the wave-length of the sinuosities on the surface of separation, the more rapidly are they magnified.

The application of this result to sensitive jets would lead us to the conclusion that their sensitiveness increases indefinitely with pitch. It is true that, in the case of certain flames, the pitch of the most efficient sounds is very high, not far from the upper limit of human hearing; but there are other kinds of sensitive jets on which these high sounds are without effect, and which require for their excitation a moderate or even a grave pitch.

A probable explanation of the discrepancy readily suggests itself. The calculations are founded upon the supposition that the changes of velocity are discontinuous—a supposition which cannot possibly agree with reality. In consequence of fluid friction, a surface of discontinuity,

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even if it could ever be formed, would instantaneously disappear, the transition from the one velocity to the other becoming more and more gradual, until the layer of transition attained a sensible width. When this width is comparable with the wave-length of the sinuosity, the solution for an abrupt transition ceases to be applicable, and we have no reason for supposing that the instability would increase for much shorter wave-lengths.

In the following investigations, I shall suppose that the motion is entirely in two dimensions, parallel (say) to the plane xy, so that (in the usual notation) w is zero, as well as the rotations ξ, η. The rotation ζ parallel to z is connected with the velocities u, v by the equation

$$\zeta = \tfrac{1}{2}\left(\frac{du}{dy} - \frac{dv}{dx}\right) \quad \text{.........................(2).}$$

When the phenomena under consideration are such that the compressibility may be neglected, the condition that no fluid is anywhere introduced or abstracted, gives

$$\frac{du}{dx} + \frac{dv}{dy} = 0 \quad \text{.............................(3).}$$

In the absence of friction, ζ remains constant for every particle of the fluid; otherwise, if ν be the kinematic viscosity, the general equation

for ζ is
$$\frac{\delta\zeta}{\delta t} = \xi\frac{dw}{dx} + \eta\frac{dw}{dy} + \zeta\frac{dw}{dz} + \nu\nabla^2\zeta \quad \text{.................(4),*}$$

where
$$\frac{\delta}{\delta t} = \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} + w\frac{d}{dz} \quad \text{.......................(5),}$$

and
$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} \quad \text{..........................(6).}$$

For the proposed applications to motion in two dimensions, these equations reduce to
$$\frac{\delta\zeta}{\delta t} = \nu\nabla^2\zeta \quad \text{.................................(7),}$$

$$\frac{\delta}{\delta t} = \frac{d}{dt} + u\frac{d}{dx} + v\frac{d}{dy} \quad \text{.........................(8),}$$

$$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2} \quad \text{.................................(9),}$$

while the two other equations similar to (4) are satisfied identically.

In order to investigate the influence of friction on stratified motion, we may now suppose that v is zero, while u and ζ are functions of y only. Our equations then give simply

$$\frac{d\zeta}{dt} = \nu\frac{d^2\zeta}{dy^2} \quad \text{............................(10),}$$

* Lamb's Motion of Fluids, p. 243.

which shows that the rotation ζ is conducted according to precisely the same laws as heat. In the case of air at atmospheric pressure, the value of ν is, according to Maxwell's experiments,

$$\nu = \cdot 16,^*$$

not differing greatly from the number (·22) corresponding to the conduction of temperature in *iron*.

The various solutions of (10), discovered by Fourier, are at once applicable to our present purpose. In the problem already referred to, of a surface of discontinuity $y = 0$, separating portions of fluid moving with different but originally constant velocities, the rotation is at first zero, except upon the surface itself, but it is rapidly diffused into the adjacent fluid. At time t its value at any point y is

$$\zeta = \int_{-\infty}^{+\infty} \zeta\, dy \cdot \frac{e^{-\frac{y^2}{4\nu t}}}{2\sqrt{(\pi\nu t)}} \qquad (11),$$

and

$$\int_{-\infty}^{+\infty} \zeta\, dy = \tfrac{1}{2} \int \frac{du}{dy}\, dy = \tfrac{1}{2}(V_2 - V_1) \qquad (12),$$

if V_2, V_1 are the velocities on the positive and negative sides of the surface respectively. If $y^2 = 4\nu t$, the value of ζ is less than that to be found at $y = 0$, in the ratio $e : 1$. Thus, after a time t, the thickness of the layer of transition $(2y)$ is comparable in magnitude with $1{\cdot}6\sqrt{t}$; for example, after one second it may be considered to be about $1\frac{1}{2}$ centimetres. In the case of water, the coefficient of conductivity is much less. It seems that $\nu = \cdot 014$; so that, after one second, the layer is about half a centimetre thick.

The circumstances of a two-dimensional jet will be represented by supposing the velocity to be limited initially to an infinitely thin layer at $y = 0$. It is convenient here to use the velocity u itself instead of ζ.

Since $\zeta = \frac{1}{2}\frac{du}{dy}$,
$$\frac{du}{dt} = \nu \frac{d^2u}{dy^2} \qquad (13),$$

and thus the solution is of the same form as before:

$$u = \int_{-\infty}^{+\infty} u\, dy \cdot \frac{e^{-\frac{y^2}{4\nu t}}}{2\sqrt{(\pi\nu t)}} \qquad (14).$$

We may conclude that, however thin a jet of air may be initially, its thickness after one second is comparable with $1\frac{1}{2}$ centimetres. A similar calculation may be made for the case of a linear jet, whose whole velocity is originally concentrated in one line.

* The centimetre and second being units.

There is, therefore, ample foundation for the opinion that the phenomena of sensitive jets may be greatly influenced by fluid friction, and deviate materially from the results of calculations based upon the supposition of discontinuous changes of velocity. Under these circumstances, it becomes important to investigate the character of the equilibrium of stratified motion in cases more nearly approaching what is met with in practice. Fully to include the effects of friction, would immensely increase the difficulties of the problem. For the present, at least, we must treat the fluid as frictionless, and be satisfied if we can obtain solutions for laws of stratification, which are free from discontinuity. For the undisturbed motion, the component velocity v is zero, and u is a function of y only. A curve in which u is ordinate and y is abscissa, represents the law of stratification, and may be called, for brevity, the velocity curve.

A class of problems which can be dealt with by fairly simple methods, is obtained by supposing the rotation ζ to be constant throughout layers of finite thickness, and only to change its value in passing a limited number of planes for which y is constant. In such cases, the velocity curve is composed of portions of straight lines which meet one another at finite angles. This state of things may be supposed to be slightly disturbed by bending the surfaces of transition, and the determination of the subsequent motion depends upon that of the form of these surfaces. For ζ retains its constant value throughout each layer unchanged in the absence of friction, and, by a well-known theorem, the whole motion depends upon ζ. We shall suppose that the functions deviate from their equilibrium values by quantities proportional to $e^{i\kappa x}$, so that everything is periodic with respect to x in a distance λ equal to $2\pi\kappa^{-1}$. By Fourier's theorem, the solution may be generalised sufficiently to cover the case of an arbitrary deformation of the surfaces. As functions of the time, the disturbances will be assumed to be proportional to e^{int}, where n may be either real or complex, and the character of the resulting motion is determined in great measure by the value of n, found by the solution of the problem.

By a theorem due to Helmholtz, the effect of any element dA rotating with angular velocity ζ, is to produce, at a point whose distance from the element is r, a transverse velocity q, such that

$$q = \frac{\zeta dA}{\pi r} \quad \text{...(15).}$$

In the application of this result to the problems in hand, it will be convenient to regard the actual value of ζ at time t as made up of two parts, (1) its undisturbed value, (2) the difference between its actual value and (1). The effect of (1) is to produce the undisturbed system of velocities, on which the small effect of (2) is superposed; and the calculation of the latter effects evidently involves integrations which extend only over the infinitely small areas included between the disturbed and undisturbed surfaces of transition. Suppose that the equation of one of these surfaces, reckoned from its undisturbed position, is

$$\eta = He^{i\kappa\xi} \qquad\qquad (16),$$

in which H is not necessarily real. Then $dA = \eta\, d\xi$, and if $\Delta\zeta$ be the excess of the value of ζ on the upper above that on the lower side of the surface, we get, by (15), at any point whose abscissa is x and distance from the surface is b,

$$q = -\frac{\Delta\zeta \,.\, \eta\, d\xi}{\pi r} \qquad\qquad (17),$$

where

$$r^2 = b^2 + (\xi - x)^2 \qquad\qquad (18).$$

The velocity q, given by (17), is perpendicular to r. The next step, previous to integration, is to resolve it in the fixed directions of x and y. The resolution is effected by introduction of the factors b/r, and $(\xi - x)/r$; and thus, for the whole effect of the surface under consideration,

$$u = -\frac{b\Delta\zeta}{\pi}\int_{-\infty}^{+\infty}\frac{\eta\, d\xi}{r^2}, \quad v = -\frac{\Delta\zeta}{\pi}\int_{-\infty}^{+\infty}\frac{\eta\,(\xi - x)\, d\xi}{r^2};$$

or, by (16),

$$u = -\frac{bH\Delta\zeta}{\pi}\int_{-\infty}^{+\infty}\frac{e^{i\kappa\xi}\, d\xi}{b^2 + (\xi - x)^2} \qquad\qquad (19),$$

$$v = -\frac{H\Delta\zeta}{\pi}\int_{-\infty}^{+\infty}\frac{e^{i\kappa\xi}\,(\xi - x)\, d\xi}{b^2 + (\xi - x)^2} \qquad\qquad (20).$$

The integrals are readily evaluated by the theorem

$$\int_{-\infty}^{+\infty}\frac{\cos ax}{1 + x^2}\,dx = \int_{-\infty}^{+\infty}\frac{x \sin ax}{1 + x^2}\,dx = \pi e^{-a};$$

and we obtain

$$\left.\begin{aligned} u &= -\ \ H\Delta\zeta\, e^{i\kappa x} e^{-\kappa b} \\ v &= -\, iH\Delta\zeta\, e^{i\kappa x} e^{-\kappa b} \end{aligned}\right\} \qquad\qquad (21).$$

In the derivation of (21), η has been treated as infinitesimal in comparison with b, but in the sequel we shall require to apply the formula to points situated upon the surface itself. The value of u would need more careful examination, but that of v is easily seen to be equally applicable when b is zero, since the neighbouring elements do not contribute sensibly to the value of the integral. In fact, the value is the same on whichever side of the surface the point under consideration is situated, and b is in both cases to be taken positive. Accordingly, when b is zero, we are to take simply

$$v = -\, iH\Delta\zeta e^{i\kappa x} \qquad\qquad (22).$$

We are now prepared to enter upon the consideration of special problems. As a first example, let us suppose that on the upper side of a layer of thickness b the undisturbed velocity u is equal to $+V$, and on the lower side to $-V$, while inside the layer it changes uniformly. Thus

$$\zeta = \tfrac{1}{2}\frac{du}{dy} = \frac{V}{b} \quad \text{.......................(23)}$$

inside the layer, and outside the layer $\zeta = 0$. In the disturbed motion, let the equations of the upper and lower surfaces be respectively, at time t,

$$\left.\begin{aligned}\eta &= He^{int}e^{i\kappa x}\\ \eta' &= H'e^{int}e^{i\kappa x}\end{aligned}\right\} \quad \text{.......................(24);}$$

then, by (21), (22), (23), the whole value of v for a point on the upper surface is

$$v = ib^{-1}Ve^{int}e^{i\kappa x}(H - H'e^{-\kappa b}) \quad \text{.......................(24)}',$$

and for the lower surface

$$v = ib^{-1}Ve^{int}e^{i\kappa x}(He^{-\kappa b} - H') \quad \text{.......................(25).}$$

From these values of v the position of the surfaces at time $t+dt$ may be calculated. At time t, η corresponds to x; at time $t+dt$, $\eta+vdt$ corresponds to $x+udt$, u being the whole component velocity parallel to x. Thus, at time $t+dt$, corresponding to x, we have

$$\eta + vdt - \frac{d}{dx}(\eta + vdt) \,.\, udt\,;$$

or, on neglecting the squares of small quantities,

$$\eta + \left(v - V\frac{d\eta}{dx}\right)dt.$$

Now, from (24),

$$\frac{d\eta}{dt} = in\eta\,;$$

so that, equating the two values of $\frac{d\eta}{dt}$, we get, from (24),

$$inH = ib^{-1}V(H - H'e^{-\kappa b}) - i\kappa VH,$$

or

$$\left(\frac{nb}{V} - 1 + \kappa b\right)H + e^{-\kappa b}H' = 0 \quad \text{.......................(26).}$$

In like manner, by considering the motion of the lower surface, we get

$$e^{-\kappa b}H + \left(-\frac{nb}{V} - 1 + \kappa b\right)H' = 0 \quad \text{.......................(27).}$$

By eliminating the ratio $H' : H$ between (26) and (27), we obtain, as the equation giving the admissible values of n,

$$n^2 = \frac{V^2}{b^2}\{(\kappa b - 1)^2 - e^{-2\kappa b}\} \quad \text{.......................(28).}$$

When κb is small, that is, when the wave-length is great in comparison with b, the case approximates to that of a sudden transition. Thus

$$n^2 = \frac{V^2}{b^2}\{1-2\kappa b+\kappa^2 b^2-(1-2\kappa b+2\kappa^2 b^2+\dots)\}$$
$$= -\kappa^2 V^2 \text{ approximately} \quad\text{.........................(29)},$$

in agreement with equation (30) of my former paper. In this case the motion is unstable. On the other hand, when κb is great, we find, from (28), $$n^2 = \kappa^2 V^2 \quad\text{.........................(30)};$$

and, since the values of n are real, the motion is *stable*. It appears, therefore, that so far from the instability increasing indefinitely with diminishing wave-length, as when the transition is sudden, a diminution of wave-length below a certain value entails an instability which gradually decreases, and is finally exchanged for actual stability. The following table exhibits more in detail the progress of $b^2V^{-2}n^2$ as a function of κb :—

κb	$b^2V^{-2}n^2$	κb	$b^2V^{-2}n^2$
·2	−·03032	1·0	−·13534
·4	−·08933	1·2	−·05072
·6	−·14120	1·3	+·01573
·8	−·16190	2·0	+·98168

We see that the instability is greatest when κb = ·8 nearly, that is, when $\lambda = 8b$; and that the passage from instability to stability takes place when $\kappa b = 1·3$ nearly, or when $\lambda = 5b$.

Corresponding with the two values of n, there are two ratios of $H' : H$ determined by (26) or (27), each of which gives a normal mode of disturbance, and by means of these normal modes the results of an arbitrary displacement of the two surfaces may be represented. It will be seen that for the stable disturbances the ratio $H' : H$ is real, indicating that the sinuosities of the two surfaces are at every moment in the same phase.

We may next take an example of a jet of thickness $2b$ moving in still fluid, supposing that the velocity in the middle of the jet is V, and that it falls uniformly to zero on either side. Taking the middle line as axis of x, we may write

$$u = V\left(1 \mp \frac{y}{b}\right) \quad\text{.........................(31)},$$

in which the − sign applies to the upper, and the + sign to the lower half of the jet. Thus $$\zeta = \tfrac{1}{2}\frac{du}{dy} = \mp\tfrac{1}{2}\frac{V}{b} \quad\text{.........................(32)}$$

within the jet, and outside the jet $\zeta = 0$. In this problem there are three surfaces to be considered. We will suppose the equation of the upper surface, for which $\Delta\zeta = \frac{V}{2b}$, to be $\eta = He^{i\kappa x}e^{int}$; that of the middle surface, for which $\Delta\zeta = -\frac{V}{b}$, to be $\eta' = H'e^{i\kappa x}e^{int}$; and that of the lower surface, for which $\Delta\zeta = \frac{V}{2b}$, to be $\eta'' = H''e^{i\kappa x}e^{int}$.

From (22), the velocities v are to be calculated as before. We find

$$v\,(\text{upper surface}) = \frac{iV}{2b}\,e^{int}e^{i\kappa x}\{-H + 2e^{-\kappa b}H' - e^{-2\kappa b}H''\},$$

$$v\,(\text{middle surface}) = \frac{iV}{2b}\,e^{int}e^{i\kappa x}\{-e^{-\kappa b}H + 2H' - e^{-\kappa b}H''\},$$

$$v\,(\text{lower surface}) = \frac{iV}{2b}\,e^{int}e^{i\kappa x}\{-e^{-2\kappa b}H + 2e^{-\kappa b}H' - H''\}.$$

For the upper and lower surfaces the horizontal velocity is zero, and for the middle surface it is V. In the same manner as for (26), we thus obtain

$$\left.\begin{aligned} mH - 2\gamma H' + \gamma^2 H'' &= 0 \\ \gamma H + (m + 2\kappa b - 3)\,H' + \gamma H'' &= 0 \\ \gamma^2 H - 2\gamma H' + mH'' &= 0 \end{aligned}\right\} \quad\text{..................(33)},$$

in which γ is written for $e^{-\kappa b}$, and m is written $2bV^{-1}n + 1$. The elimination from (33) of $H : H' : H''$ gives the following cubic in m :—

$$m^3 + (2\kappa b - 3)\,m^2 + \gamma^2(4 - \gamma^2)\,m - \gamma^4(1 + 2\kappa b) = 0 \quad\text{............(34)}.$$

By inspection of (33), we see that one of the normal disturbances is defined by $H' = 0$, $H + H'' = 0$, and that the corresponding value of m is γ^2. It follows that $m - \gamma^2$ is a factor of the cubic expression in (34), and the remaining quadratic factor is readily obtained by division. Thus (34) assumes the form

$$(m - \gamma^2)\{m^2 + (2\kappa b - 3 + \gamma^2)\,m + \gamma^2(1 + 2\kappa b)\} = 0 \quad\text{.........(35)}.$$

For the symmetrical disturbance

$$n = -\frac{V}{2b}\,(1 - e^{-2\kappa b}) \quad\text{.............................(36)},$$

a *real* quantity, indicating that the motion is *stable* so far as this mode of disturbance is concerned.

The other two values of n are real, if

$$(2\kappa b - 3 + \gamma^2)^2 - 4\gamma^2(1 + 2\kappa b) \quad\text{.......................(37)}$$

be positive, but not otherwise. When κb is infinite, $\gamma = 0$, and (37) reduces to $4\kappa^2 b^2$, which is positive; so that the motion is stable when the wave-length is small in comparison with the thickness of the jet. On the other hand, as may readily be proved by expanding γ in (37), the motion is unstable, when the wave-length is great in comparison with the thickness of the jet. The values of (37) can be more easily computed when it is thrown into the form

$$(5+2\kappa b-e^{-2\kappa b})^2-16\,(1+2\kappa b) \quad\ldots\ldots\ldots\ldots(38).$$

Some corresponding values of (38) and $2\kappa b$ are shown below :—

$2\kappa b$	(38)	$2\kappa b$	(38)
·5	−·054	2·5	−·975
1·0	−·279	3·0	−·794
1·5	−·599	3·5	−·263
2·0	−·876	4·0	+·671

The wave-length of maximum instability is about $2\frac{1}{2}$ times the thickness ($2b$) of the jet; while, for a wave-length about half as great again, or more, the motion becomes stable.

Although it is the fact, as I have found by experiment, that a sensitive jet breaks up by becoming sinuous as a whole, the result that a symmetrical mode of disturbance is stable, is special to the law of velocity assumed in the foregoing example. In order to illustrate this, I will state the results for the more general law of velocity obtained by supposing the maximum velocity V to extend through a layer of finite thickness b' in the middle of the jet. The rotation ζ is zero in this central layer; in the adjacent layers of thickness b, $\zeta = \mp \frac{V}{2b}$, as before. The equations of the four surfaces, in crossing which ζ changes its value, being represented by

$$\eta,\ \eta',\ \eta'',\ \eta''' = (H,\ H',\ H'',\ H''')\, e^{int} e^{i\kappa x},$$

we may obtain four equations involving n and the three ratios $H : H' : H'' : H'''$. The elimination of these ratios gives a biquadratic in n, which, however, is easily split into two quadratics, one of which relates to symmetrical disturbances, for which $H+H'''=0$, $H'+H''=0$; and the other to disturbances for which $H-H'''=0$, $H'-H''=0$. The resulting equation in n is

$$\left(\frac{2bn}{V}\right)^2+(\pm\gamma'\mp\gamma'\gamma^2+2\kappa b)\,\frac{2bn}{V}$$
$$\pm\gamma'-1+2\kappa b+\gamma^2\,(1\mp\gamma'\mp 2\kappa b\gamma')=0 \quad\ldots\ldots(39),$$

γ' being written for $e^{-\kappa b'}$. In (39) the upper signs correspond to the symmetrical displacements. The roots are real, and the disturbances are stable, if

$$(\pm\gamma'\mp\gamma'\gamma^2+2\kappa b)^2-4\,[\pm\gamma'-1+2\kappa b+\gamma^2\,(1\mp\gamma'\mp 2\kappa b\gamma')] \quad\ldots(40)$$

be positive.

In what follows, we will limit our attention to the symmetrical disturbances, that is, to the upper signs in (40), and to terms of orders not higher than the first in b'. The expression (40) may then be reduced to $(1-\gamma^2-2\kappa b)^2+2\kappa b'(1+\gamma^2)(1-\gamma^2-2\kappa b)$ ....(41).

If κb be very small, this becomes

$$4\kappa^4b^4-8\kappa b'.\kappa^2b^2 \quad\quad (42).$$

If b' be zero, (42) is positive, and the disturbance is stable, as we found before; but if b and b' be of the same order of magnitude, and both very small compared to λ, it follows from (42) that the disturbance is unstable.

If, in (39), we suppose that b is zero, we fall back upon the case of a jet of uniform velocity V and thickness b' moving in still fluid. The equation for n, after division by b^2, becomes

$$n^2+(1\pm\gamma')\kappa V.n+\tfrac{1}{2}(1\pm\gamma')\kappa^2V^2=0,$$

or

$$(n+\kappa V)^2\frac{1\pm\gamma'}{1\mp\gamma'}+n^2=0 \quad\quad (43)$$

In the notation of my former paper, $b'=\tfrac{1}{2}l$, so that

$$\frac{1+\gamma'}{1-\gamma'}=\coth\kappa l,\quad \frac{1-\gamma'}{1+\gamma'}=\tanh\kappa l;$$

and the equations there numbered (48) and (55) agree with (43).

Another particular case of (39), comparable with previous results, is obtained by supposing b' to be infinite.

I now pass to the consideration of certain cases in which the moving layers are bounded by fixed walls, instead of by an unlimited expanse of stationary fluid. The effect of the walls may be imitated by the introduction of an unlimited number of similar layers, in the same way as the vibrations of a string fixed at two points are often deduced from the theory applicable to an unlimited string. The displacements of the surfaces at which ζ changes its value being taken equal and opposite in consecutive layers, the value of v, at the places occupied by the walls, is, by symmetry, zero; and thus the presence or absence of the actual walls is a matter of indifference.

Let us first suppose that the distribution of velocity within the layer is that given in (31), uniformly increasing from zero at the walls to a maximum V in the middle, the distance between the walls being $2b$. The actual surface of transition and its successive images make contributions to the value of v at the surface, which are alternately opposite in sign, and, as regards numerical value, form a geometrical progression with common ratio $e^{-2\kappa b}$ or γ^2. Thus, with the same notation as before, we have at the surface, from (21), (22),

$$v = \frac{iVH}{b} e^{int} e^{i\kappa x} \{1-2(\gamma^2-\gamma^4+\gamma^6-\dots)\}$$

$$= \frac{iVH}{b} e^{int} e^{i\kappa x} \cdot \frac{1-\gamma^2}{1+\gamma^2};$$

so that, as in previous problems,

$$n = -\kappa V + \frac{V}{b}\,\frac{1-\gamma^2}{1+\gamma^2};$$

or, as it may also be written,

$$\frac{bn}{V} = \tanh \kappa b - \kappa b \text{ ....(44).}$$

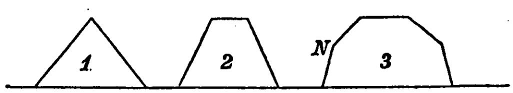
If there be a layer of finite width b' in the centre, throughout which the undisturbed velocity is V, we obtain

$$\left.\begin{aligned} \left(2\frac{nb}{V}+2\kappa b-\frac{1-\gamma^2\gamma'}{1+\gamma^2\gamma'}\right)H-\frac{\gamma'-\gamma^2}{1+\gamma^2\gamma'}H'=0 \\ -\frac{\gamma'-\gamma^2}{1+\gamma^2\gamma'}H+\left(2\frac{nb}{V}+2\kappa b-\frac{1-\gamma^2\gamma'}{1+\gamma^2\gamma'}\right)H'=0 \end{aligned}\right\} \text{ ....(45),}$$

in which H, H' refer to the two surfaces of transition, and $\gamma' = e^{-\kappa b'}$. Equation (45) shews, as might also be inferred from symmetry, that $H \pm H' = 0$, while

$$\frac{2nb}{V}+2\kappa b-\frac{1-\gamma^2\gamma'}{1+\gamma^2\gamma'} = \pm\frac{\gamma'-\gamma^2}{1+\gamma^2\gamma'} \text{ ....(46).}$$

Since the values of n in (46) are real, the disturbance is *stable*.



In these examples the velocity curves are those represented by figs. (1) and (2). I have taken a further step in the direction of generalisation by calculating the motion for a velocity curve in the form of (3). The criterion of stability is complicated in its expression, but it is not difficult to shew that the motion is stable if the angle N be a projecting angle. From these examples there seemed to be some reason for thinking that the motion would be stable, whenever the velocity curve was of one curvature throughout; and this led me to attack the question by a more general method, which I will now explain.

Let us suppose that the conditions of steady motion are satisfied by $u = U$, $v = V$, $\zeta = Z$; and let us trace the effects of superposing upon this motion a disturbance for which $u = \delta u$, $v = \delta v$, $\zeta = \delta\zeta$. Both the original motion and the disturbance satisfy the equation of continuity (3).

Since, in the absence of friction, the rotation of every element remains unchanged, $$\frac{d(Z+\delta\zeta)}{dt}=0,$$

or $$\frac{d}{dt}(Z+\delta\zeta)+(U+\delta u)\frac{d}{dx}(Z+\delta\zeta)+(V+\delta v)\frac{d}{dy}(Z+\delta\zeta)=0.$$

This equation is satisfied by supposition, if δu, δv, $\delta\zeta=0$. If we omit the squares and products of the small quantities, it becomes

$$\frac{d\delta\zeta}{dt}+U\frac{d\delta\zeta}{dx}+V\frac{d\delta\zeta}{dy}+\delta u\frac{dZ}{dx}+\delta v\frac{dZ}{dy}=0 \quad\text{.............}(47).$$

If $V=0$, and U be a function of y only, (47) reduces to

$$\frac{d\delta\zeta}{dt}+U\frac{d\delta\zeta}{dx}+dv\frac{dZ}{dy}=0;$$

or, since in this case $Z=\frac{1}{2}\frac{dU}{dy}$,

$$\left(\frac{d}{dt}+U\frac{d}{dx}\right)\left(\frac{d\delta u}{dy}-\frac{d\delta v}{dx}\right)+\frac{d^2U}{dy^2}\delta v=0 \quad\text{.............}(48).$$

We now introduce the supposition that, as functions of x, δu and δv are proportional to $e^{i\kappa x}$, so that, by (3),

$$i\kappa\delta u+\frac{d\delta v}{dy}=0 \quad\text{..................................}(49).$$

We thus obtain, by elimination of δu,

$$\left(\frac{1}{i\kappa}\frac{d}{dt}+U\right)\left(\frac{d^2\delta v}{dy^2}-\kappa^2\delta v\right)-\frac{d^2U}{dy^2}\delta v=0 \quad\text{...............}(50).$$

If we further suppose that, as a function of t, δv is proportional to e^{int}, where n is a real or complex constant, we get

$$\left(\frac{n}{\kappa}+U\right)\left(\frac{d^2\delta v}{dy^2}-\kappa^2\delta v\right)-\frac{d^2U}{dy^2}\delta v=0 \quad\text{..................}(51).$$

On this equation the solution of the special problems already considered may be founded. If, throughout any layer, the rotation Z be constant, $\frac{d^2U}{dy^2}=0$, and, wherever $n+\kappa U$ is not equal to zero, (51) reduces to

$$\frac{d^2\delta v}{dy^2}-\kappa^2\delta v=0 \quad\text{..................................}(52).$$

Equation (52) may, in fact, be easily established independently, on the assumption that the rotation throughout the layer is the same after disturbance as before. From (2),

$$\frac{d\zeta}{dx}=\frac{1}{2}\left(\frac{d^2u}{dx\,dy}-\frac{d^2v}{dx^2}\right)=-\frac{1}{2}\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}\right),$$

by (3); so that, when ζ is constant, $\nabla^2 v = 0$. In like manner $\nabla^2 u = 0$. If $\delta v \propto e^{i\kappa x}$, (52) follows immediately.

The solution of (52) is

$$\delta v = Ae^{\kappa y} + Be^{-\kappa y} \quad\text{.................................(53),}$$

where A and B are constants, not restricted to be real. For each layer of constant ζ, a fresh solution with fresh arbitraries is to be taken, and the partial solutions are to be fitted together by means of the proper boundary conditions. The first of these conditions is evidently

$$\Delta \delta v = 0 \quad\text{.......................................(54).}$$

The second may be obtained by integrating (51) across the boundary. Thus

$$\left(\frac{n}{\kappa} + U\right) . \Delta\left(\frac{d\delta v}{dy}\right) - \Delta\left(\frac{dU}{dy}\right) . \delta v = 0 \quad\text{..................(55).}$$

At a fixed wall $\delta v = 0$.

The reader may apply this method to the problem whose solution is expressed in (44).

In cases where $\frac{d^2U}{dy^2} = 0$, the substitution of (52) for (51), or the corresponding supposition that ζ is unchanged by the disturbance, amounts to a limitation on the generality of the solution. Suppose, for example, that the motion takes place between two fixed walls, at each of which $\delta v = 0$. Under these circumstances (53) shews that $\delta v = 0$ throughout, or no disturbance is possible; and this is obviously true if no new rotation is introduced by the disturbance. In order to obtain a general solution, we must retain the factor $n + \kappa U$ in (51). For any value of y which gives $n + \kappa U = 0$, (52) need not be satisfied; and thus any value of $-\kappa U$ is an admissible value of n, satisfying all the conditions of the problem.

I will now inquire, under what conditions (51) admits of a solution with a complex value of n; or, in other words, under what conditions the steady motion is unstable, assuming that, for two finite or infinite values of y, $\delta v = 0$. Let $n \div \kappa = p + iq$, $\delta v = \alpha + i\beta$, where p, q, α, β are real. Substituting in (51), we get

$$\frac{d^2\alpha}{dy^2} + i\frac{d^2\beta}{dy^2} = \left[\kappa^2 + \frac{d^2U}{dy^2}\,\frac{p + U - iq}{(p+U)^2 + q^2}\right](\alpha + i\beta);$$

or, on equating to zero the real and imaginary parts,

$$\frac{d^2\alpha}{dy^2} = \kappa^2\alpha + \frac{d^2U}{dy^2}\,\frac{(p+U)\,\alpha + q\beta}{(p+U)^2 + q^2} \quad\text{..........................(56),}$$

$$\frac{d^2\beta}{dy^2} = \kappa^2\beta + \frac{d^2U}{dy^2}\,\frac{-q\alpha + (p+U)\,\beta}{(p+U)^2 + q^2} \quad\text{.......................(57).}$$

Multiplying (56) by β, (57) by α, and subtracting, we get

$$\beta\frac{d^2\alpha}{dy^2}-\alpha\frac{d^2\beta}{dy^2}=\frac{d^2U}{dy^2}\frac{q(\alpha^2+\beta^2)}{(p+U)^2+q^2}=\frac{d}{dy}\left(\beta\frac{d\alpha}{dy}-\alpha\frac{d\beta}{dy}\right)\ldots\ldots(58).$$

At the limits δv, and therefore both α and β, are, by hypothesis, zero. Hence, integrating (58) between the limits, we see that q must be zero, if $\frac{d^2U}{dy^2}$ be of one sign throughout the range of integration; so that, if the velocity curve is either wholly convex or wholly concave for the space between two limits at which $\delta v=0$, the motion is thoroughly stable.* This result covers all the special problems of motion between walls previously investigated. Its application to jets, for which $\frac{d^2U}{dy^2}$ changes sign, leaves the question of stability or instability still open.

Another general result worth notice may be obtained from (51). Writing it in the form

$$\frac{d^2\delta v}{dy^2}=\left\{\kappa^2+\frac{\frac{d^2U}{dy^2}}{\frac{n}{\kappa}+U}\right\}\delta v,$$

we see that, if n is real, δv cannot pass from one zero value to another zero value, unless $\frac{d^2U}{dy^2}$ and $n+\kappa U$ be somewhere of contrary signs. Thus, if we suppose that U is positive, and $\frac{d^2U}{dy^2}$ negative throughout, and that V is the greatest value of U, we find that $n+\kappa V$ must be positive. For an example see the equation immediately preceding (44).

If the stream lines of the steady motion be concentric circles instead of parallel straight lines, the character of the problem is not greatly changed. It may be proved that, if the fluid move between two rigid concentric circular walls, the motion is stable, provided that in the steady motion the rotation either continually increases or continually decreases in passing outwards from the axis.

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* More generally, the same conclusion follows if the ratio $\frac{d\delta v}{dy}:\delta v$ has real values at both limits.

March 11*th*, 1880.

C. W. MERRIFIELD, Esq., F.R.S., President, in the Chair.

The following gentlemen were elected members:—

Messrs. C. S. Peirce, Lecturer on Logic, Johns Hopkins University; Emery McClintock, Milwaukee, Wisconsin; Prof. E. B. Seitz, M.A., Kirksville, Missouri; and Mr. E. Temperley, M.A., Queens' College, Cambridge. Mr. W. J. Curran Sharp was admitted into the Society. Dr. Elling Holst, of Christiania, was present at the meeting.

The following communications were made:—

"Notes on a General Method of solving Partial Differential Equations of the First Order with several Dependent Variables:" Mr. H. W. Lloyd Tanner, M.A.

"Note on the Integral Solution of $x^2-2Py^2=-z^2$ or $\pm 2z^2$ in certain cases:" Mr. S. Roberts, F.R.S.

Notes (1) "On a Geometrical Form of Landen's Theorem with regard to a Hyperbolic Arc;" (2) "On a class of Closed Ovals whose axes possess the same property as two Fagnanian arcs of an Ellipse:" Mr. J. Griffiths, M.A.

The following presents were received:—

"Om Poncelet's betydning for Geometrien et bidrag til de moderngeometriske ideers udviklingshistorie," af Elling Holst; Christiania, 1878: from the Author.

"Lettres de Laplace, Borda, Fuss et Jean-Albert Euler" (Extrait du "Bulletin des Sciences Mathématiques et Astronomiques," 2e Série, Tome iii., 1879).

"De l'emploi des fonctions elliptiques dans la théorie du quadrilatère plan," par M. G. Darboux. (Ditto.)

"Addition au mémoire sur les fonctions discontinues," par M. G. Darboux ("Annales Scientifiques de l'Ecole Normale Supérieure," Paris, pp. 195—202).

"Applicazione dei Principii della Meccanica Analitica a Problemi"—Note i., ii., iii., iv., di Alessandro Dorna, Torino 1879: presented by the Meteorological Office.

"Ueber die Erhaltung des Geschlechtes bei Zwei ein-eindeutig auf einander bezogenen Plan-curven," von H. Schubert, in Hamburg ("Math. Ann.," xvi., pp. 180—182).

"Proceedings of the Royal Society," Vol. xxx., No. 201.

"Bulletin des Sciences Mathématiques et Astronomiques," Tome iii., Oct. 1879; Paris, 1879.

"Atti della R. Accademia dei Lincei—Transunti," Fasc. 3°, Febbraio 1880, Vol. iv.; Roma, 1880.

"Beiblätter zu den Annalen der Physik und Chemie," Band iv., Stück 3; Leipzig, 1880.

"Mémoire sur les solutions singulières des équations aux dérivées partielles du premier ordre," par M. G. Darboux; Paris, 1880: from the Author.

"Etudes Cinématiques," par M. E. J. Habich; Paris, 1879.

"Bulletin de la Société Mathématique de France," Tome viii., No. 1; Paris, 1880.

"Johns Hopkins University Circulars," No. 3 (Number devoted to Mathematics and Physics); Baltimore, Feb. 1880.

"Monatsbericht," Dec. 1879; Berlin, 1880.

"Crelle," Band 89, 1es Heft; Berlin, 1880.

"Proceedings of the Royal Society of Edinburgh," Session 1878—1879, Vol. x., 103, pp. 1—314.

"Three approximate solutions of Kepler's Problem," by H. A. Howe, A.M. (Cincinnati Society of Natural History, Proceedings, pp. 205—210).

"Educational Times," March, 1880.

"Ueber unendliche lineare Punktmannigfaltigkeiten," von Georg Cantor ("Math. Ann.," Bd. xv.)

"Zur Theorie der Zahlentheoretischen Functionen," von Georg Cantor.

Notes on a General Method of Solving Partial Differential Equations of the First Order with several Dependent Variables. By H. W. Lloyd Tanner, M.A.

[*Read March 11th*, 1880.]

1. The following notes relate to a method of solving equations with several dependent variables, which is a generalization of the process employed in the case where only one dependent variable occurs. The extension to the more general equations herein considered is easy, but it is perhaps worth stating since it leads to this curious conclusion; that the solution of such equations depends upon the solution of an auxiliary system similar to that which is required in the integration of equations with one dependent variable, but involving differential coefficients of the second or higher orders.

When there is one dependent variable, z, we determine p_i $\left(\text{or } \frac{dz}{dx_i}\right)$ by means of equations

$$F_1 = 0,\ \ F_2 = 0,\ \dots\ F_n = 0,$$