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The Jacobian Matrix and Global Univalence of Mappings*

By

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1. Introduction

As is well known, the classical implicit function theorem assures the local univalence of a mapping in a neighborhood of a point at which its Jacobian does not vanish, but it does not necessarily imply global univalence in a region, even if the Jacobian is everywhere non-vanishing. The purpose of this note is to give some further useful conditions on the Jacobian matrix which are sufficient to insure global univalence.

Our main result asserts that if all principal submatrices of the Jacobian matrix have positive determinants the mapping is univalent in any rectangular region. Matrices with this property are termed P-matrices and their algebraic properties are derived in Section 2, especially as regards their relation to linear inequalities. After establishing notations in Section 3, we extend the linear results of Section 2 to the non-linear situation, from which a univalence theorem is derived in Section 4.

Among the P-matrices two subclasses have received special attention, Leontief matrices and positive quasi-definite matrices (these are defined in Sections 2, 4) and for these one can prove somewhat stronger theorems. Thus, if the Jacobian of a mapping is a Leontief matrix, then the inverse mapping is monotonic (Section 4), while if the Jacobian matrix is quasi-definite, univalence obtains not only on rectangular but on any convex regions (Section 5).

In Section 6, with the aid of the Kronecker theorem on indices, the principal univalence theorems are generalized so that the conditions on the Jacobian matrix are weakened.

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The final section discusses some special two-dimensional cases.

We remark that these investigations were stimulated by an assertion by P. A. SAMUELSON [5] that univalence holds if the upper left-hand principal minors of the Jacobian do not vanish in a region. But this is not true even in rectangular regions as shown by the following example: a mapping of R^2 into itself is given by $f(x,y) = e^{2x} - y^2 + 3$, $g(x,y) = 4e^{2x}y - y^3$. Then, $f_x = 2e^{2x} > 0$,

$$\begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix} = 2e^{2x}(4e^{2x} + 5y^2) > 0 \quad \text{in} \quad R^2 .$$

However there are two points $(0, 2)$ and $(0, -2)$ which are mapped into the origin.

Among the numerous questions which are not settled here are: (1) If all principal minors are non-vanishing, does univalence obtain?; (2) If the Jacobian is not zero and all the entries are non-negative, does univalence obtain? Both (1) and (2) are answered in the affirmative in rectangular regions in R^2, but even the conjunction of (1) and (2) has not been proved sufficient in general, though it is solved in the affirmative in rectangular regions in R^3, as shown by a special argument not given here.

2. Preliminary Results on P-matrices

2.1 P-matrices. An $n \times n$ real matrix $A = (a_{ij})$ is said to be a *P-matrix*, if all its principal minors are positive. We shall be concerned with some useful results on P-matrices, which may also be of independent interest.

In the sequel frequent use will be made of a semi-order in the real n-space R^n. For $x = (x_i)$, $y = (y_i) \in R^n$, the following notations are defined:

$$x \geqq y \quad \text{if} \quad x_i \geqq y_i \quad (i = 1, 2, \ldots, n) .$$

$$x \geq y \quad \text{if} \quad x \geqq y \quad \text{and} \quad x \neq y .$$

$$x > y \quad \text{if} \quad x_i > y_i \quad (i = 1, 2, \ldots, n) .$$

A vector x is termed *nonnegative*, if $x \geqq 0$.

Theorem 1. *If A is a P-matrix, then the inequalities*

$$Ax \leqq 0, \quad x \geqq 0 \tag{1}$$

have only the trivial solution $x = 0$.

Proof. The result is immediate for $n = 1$. Assume it true in dimensions lower than n and let $x = (\xi_i)$ satisfy (1). Since A is a P-matrix, it is non-singular, and the diagonal entries of its inverse $A^{-1} = (b_{ij})$ are positive, whence any column of A^{-1}, say the first column, b certainly has some positive components. Let θ be the minimum of ξ_i/b_{i1} over all positive components of b and let this minimum be attained for $i = k$. Then $\theta \geqq 0$, $y = x - \theta b = (\eta_j) \geqq 0$ and $\eta_k = 0$. Note that $Ay = Ax - \theta\, Ab = Ax - \theta(\delta_{i1}) \leqq 0$, where δ_{ij} are the Kronecker deltas. Let $\hat{A}$ be the principal submatrix obtained from A by deleting its kth row and column, and let $\hat{y}$ be the $(n-1)$-vector obtained from y by deleting its kth component. Then, we have $\hat{A}\hat{y} \leqq 0$, $\hat{y} \geqq 0$. Since $\hat{A}$ is an $(n-1) \times (n-1)$ P-matrix, it follows, by the induction hypothesis, that

$\hat{y} = 0$. This, combined with $\eta_k = 0$, gives $y = 0$, so that $Ax = \theta(\delta_{i1}) \geqq 0$. This implies, in view of $Ax \leqq 0$, that $Ax = 0$. Whence, by the non-singularity of A, we have $x = 0$, q. e. d.

Corollary 1. *If A is a P-matrix, there is a number $\lambda > 0$ such that for all nonnegative vectors $x \geqq 0$ of norm 1 ($\|x\| = 1$) some component of Ax is as great as λ.*

Proof. Let $(\eta_i) = Ax$ and let $\eta(x) = \max_i \eta_i$. Then $\eta(x)$ is continuous and attains a minimum λ on the compact set of all nonnegative vectors of norm 1. But, by Theorem 1, λ must be positive.

Corollary 2. *If A is a P-matrix, the inequalities*

$$Ax > 0, \quad x > 0 \tag{2}$$

have a solution.

Proof. This follows by standard duality for linear inequalities, say, the theorems due to STIEMKE [6] and TUCKER [7]. In fact, since A is a P-matrix so is its transpose A'. By Theorem 1, therefore, $p'A \geqq 0$, $p' \leqq 0$ imply $p'A = 0$, $p' = 0$, which is nothing but the Stiemke condition being equivalent to the existence of positive vectors $x > 0$ and $u > 0$ such that $Ax - u = 0$. This completes the proof.

2.2 Geometric Characterization of P-matrices. Note that Theorem 1 states that a P-matrix can not map any point except zero from the positive into the negative orthant. A simple generalization of this property turns out to be both necessary and sufficient for A to be a P-matrix.

Let A be an $n \times n$ matrix, $x = (\xi_i)$ be a column vector, and let $y = (\eta_i) = Ax$. Then A is said to *reverse the sign* of x if $\xi_i \eta_i \leqq 0$ for all i.

Theorem 2. *A is a P-matrix if and only if A reverses the sign of no vector except zero.*

Proof. First, in proving necessity, we note that one has only to consider the case where $x \geqq 0$. For, if $x = (\xi_i) \ngeqq 0$, let $L = \{i \mid \xi_i < 0\}$ and let D be the diagonal matrix obtained from the identity matrix by replacing its ith rows e^i $(i \in L)$ by $-e^i$. Then, the matrix $A^* = DAD$ is again a P-matrix, since we have simply changed the signs of a set of rows and the corresponding set of columns of A. Moreover, A^* reverses the sign of $Dx \geqq 0$.

Now suppose that $x = (\xi_i) \geqq 0$ and A reverses the sign of x. Let $M = \{i \mid \xi_i > 0\}$. Assume $M \neq \emptyset$ and let $\hat{A}$ be the principal submatrix of A obtained by deleting its ith rows and columns for $i \notin M$ and let $\hat{x}$ be the corresponding vector obtained from x. Then $\hat{A}$ is again a P-matrix and reverses the sign of $\hat{x}$. As every component of $\hat{x}$ is positive, no component of $\hat{A}\hat{x}$ can be positive, so that $\hat{A}\hat{x} \leqq 0$. Whence, by Theorem 1, $\hat{x}$ must be zero, contradicting $\hat{x} > 0$. This proves necessity.

Conversely, let $A^* = (a_{ij})$, $(i, j \in M)$ be any principal submatrix of A, where M is the corresponding set of numbers of rows and columns. If the determinant of A^* were nonpositive, A^* would have one real nonpositive eigen-value and a corresponding real eigen-vector $x^* = (\xi^*) \neq 0$, since the determinant of A^* equals the product of all the eigen-values of A^* and complex eigen-values occur as pairs of conjugate complex numbers. If then we let x be the vector whose component $\xi_i = \xi^*$ for $i \in M$, $=0$ for $i \notin M$, A clearly reverses the sign of the nonzero x, arriving at a contradiction. Therefore, the determinant is positive.

2.3 Examples of P-matrices. P-matrices include as special cases two classes which occur frequently in economics: namely, (α) matrices with positive dominant diagonal, and (β) positive quasi-definite matrices. The following explicit definitions will be helpful to see this situation.

An $n \times n$ matrix A is said to have *dominant diagonal*, if there are n positive numbers $d_i > 0$ such that

$$|a_{ii}|\, d_i > \sum_{j \neq i} |a_{ij}|\, d_j \qquad (i = 1, 2, \ldots, n)\,.$$

If a matrix with dominant diagonal has positive diagonal entries, then it is a P-matrix (see [3]). An important subclass of (α) are matrices with dominant diagonal and *of the Leontief type* (see 4.4).

On the other hand, an $n \times n$ matrix A is said to be *positive quasi-definite*, if its symmetric part, namely $\frac{1}{2}(A + A')$ is positive definite. In this case, A as well as $\frac{1}{2}(A + A')$ is a P-matrix, as is well known.

3. Differentiable Mappings

3.1 Regions. A *region* is an open connected set in R^n, either without its boundary or together with its boundary. If necessary, we term the former an *open region* and the latter a *closed region* to distinguish one from the other. An *open rectangular region*, or simply an *open interval* is the set $\{x \mid p_i < x_i < q_i\ (i = 1, 2, \ldots, n)\}$, where p_i and q_i are real numbers; we allow some or all of them to be $\pm\infty$. A *closed rectangular region* is the set $\{x \mid p_i \leqq x_i \leqq q_i\ (i = 1, 2, \ldots, n)\}$, where $-\infty < p_i < q_i < +\infty$. An *arbitrary rectangular region* is the set obtained from a closed rectangular region by replacing some or all of the defining inequalities by the corresponding strict inequalities.

3.2 Differentiable Mappings. A set of n real-valued functions $f_i(x)$ defined on a region Ω gives rise to a mapping $F : \Omega \to R^n$, by the formula $F(x) = (f_i(x))$. The mapping F is said to be *differentiable* in Ω, if every component $f_i(x)$ has a total differential $\sum_{j=1}^{n} f_{ij}(x)\, dx_j$ at each point x of Ω; that is, for $x, a \in \Omega$, we have the expression

$$f_i(x) = f_i(a) + \sum_{j=1}^{n} f_{ij}(a)\,(x_j - a_j) + o(\|x - a\|) \qquad (i = 1, 2, \ldots, n)\,, \tag{3}$$

where $o(\|x - a\|)$ stands for the Landau's o-symbol.

The *Jacobian matrix* of the mapping F is denoted by J, or J_F if necessary, and defined by $J(x) = (f_{ij}(x))$. Differentiability clearly implies continuity. It also implies partial differentiability, and f_{ij} turns out to be $\partial f_i/\partial x_j$. However, it should be noted that in the case of a non-open region partial differentiation, even if one-sided, may not be carried out at some boundary points. Yet the Jacobian can be defined by means of the coefficients of the total differential. All the results below, except some of the last section, will be proved on the assumption of differentiability, without requiring continuous differentiability.

4. Case of the *P*-Jacobian Matrix

In this section we extend Theorem 1 to the non-linear case from which we will prove a univalence theorem for the case where the Jacobian matrix is a P-matrix throughout a rectangular region.

4.1 Nonlinear Extension of Theorem 1. Let us consider a differentiable mapping $F : \Omega \to R^n$, where Ω is a closed rectangular region $\{x \mid p \leqq x \leqq q\}$ of R^n.

Theorem 3. *If the Jacobian matrix $J(x)$ of the mapping F is a P-matrix at every $x \in \Omega$, then for any a and x in Ω the inequalities*

$$F(x) \leqq F(a), \quad x \geqq a \tag{4}$$

have only the solution $x = a$.

Proof. By a simple translation we may assume that $F(a) = 0$. We proceed by induction on n, the result being obvious for $n = 1$. Let X be the set of all solutions of (4). We assert first that a is an isolated point of X. In fact, differentiability implies, as was stated in (3), that

$$\lim_{x \to a} (F(x)/\|x - a\| - J(a)(x - a)/\|x - a\|) = 0\,. \tag{5}$$

Since $J(a)$ is a P-matrix by Corollary 1 the second term above has some component greater than some positive constant for all $x \geq a$. Then, (5) implies that in some neighborhood of a, $F(x)$ has at least one positive component for x with $x \geq a$. Hence (4) with $F(a) = 0$ is satisfied only by a in this neighborhood.

Now let $\hat{X} = X - \{a\}$. From the preceding discussion $\hat{X}$ is closed and therefore compact. If $\hat{X}$ were not empty, it therefore should contain a minimal element $\bar{x}$ with the property that no other element x of $\hat{X}$ satisfies $x \leq \bar{x}$. We now distinguish two cases.

Case 1. $\bar{x} > a$. Since $J(\bar{x})$ is a P-matrix, by Corollary 2 there is a vector $u < 0$ such that $J(\bar{x})u < 0$. Because $\bar{x} > a$ we can choose λ positive but so small that $x(\lambda) = \bar{x} + \lambda u > a$. Thus $a < x(\lambda) < \bar{x}$ so $x(\lambda)$ lies in Ω. Further, since F is differentiable, we have

$$F(x(\lambda)) = F(\bar{x}) + \lambda J(\bar{x})u + o(\lambda\|u\|)$$

so that

$$\frac{F(x(\lambda)) - F(\bar{x})}{\lambda\|u\|} - J(\bar{x})\frac{u}{\|u\|}$$

can be made as small as we like by choosing further smaller positive values λ. This implies that $F(x(\lambda)) < F(\bar{x}) \leqq F(a)$ and $x(\lambda) \in \Omega$ for a sufficiently small positive λ, contradicting the minimality of $\bar{x}$. Hence $\hat{X} = \emptyset$.

Case 2. Some component of $\bar{x} = (\xi_i)$ is equal to the corresponding component of $a = (\alpha_i)$. By a simple identical renumbering of the equations and variables we may assume that the first component ξ_1 of $\bar{x}$ equals α_1, the corresponding one of a. Then define a new mapping $\hat{F} : \hat{\Omega} \to R^{n-1}$ by the formula

$$\hat{f}_i(x_2, \ldots, x_n) = f_i(\alpha_1, x_2, \ldots, x_n) \qquad (i = 2, \ldots, n)$$

where $\hat{\Omega} = \{(x_2, \ldots, x_n) \mid p_i \leqq x_i \leqq q_i \ (i = 2, \ldots, n)\}$. The Jacobian matrix of this new mapping is again a P-matrix, and clearly $\hat{f}_i(\alpha_2, \ldots, \alpha_n) = 0 \geqq \geqq \hat{f}_i(\xi_2, \ldots, \xi_n)$ $(i = 2, \ldots, n)$. So by the induction hypothesis $\alpha_i = \xi_i$ $(i = 2, \ldots, n)$, and therefore $\bar{x} = a$, contrary to the assumption $a \notin \hat{X}$. This completes the proof.

4.2 Proof of a Univalence Theorem. We are now ready to prove

Theorem 4. *If $F : \Omega \to R^n$, where Ω is a closed rectangular region of R^n, is a differentiable mapping such that the Jacobian matrix $J(x)$ is a P-matrix for all x in Ω, then F is univalent in Ω.*

Proof. Suppose $a, b \in \Omega$ and $F(a) = F(b)$. We must show that $a = b$. Letting $a = (\alpha_i)$, $b = (\beta_i)$ we may suppose, reordering if necessary, that

$$\alpha_i \leqq \beta_i (i \leqq k), \quad \alpha_i \geqq \beta_i \, (i > k) \,. \tag{6}$$

Then, if $k = n$, we observe that $F(a) = F(b)$ and $a \leqq b$; that is, the conditions of Theorem 3 are met. Hence, by the theorem, we have $a = b$. The case $k = 0$ can be managed likewise. If $0 < k < n$, define the mapping $D : R^n \to R^n$ by

$$D(x_1, \ldots, x_n) = (x_1, \ldots, x_k, -x_{k+1}, \ldots, -x_n) \,.$$

Then D is univalent on R^n and $D^{-1} = D$. Further $D(\Omega)$ is again a closed rectangular region. Let $D(a) = a^*$ and $D(b) = b^*$. Finally let $H : D(\Omega) \to R^n$ be the composite mapping given by $H = D \circ F \circ D$. One verifies that $H(a^*) = H(b^*)$ and $a^* \leqq b^*$. Moreover the Jacobian matrix of H is a P-matrix since it is obtained from that of F by changing row and column signs in the same way as in Theorem 2. Hence by Theorem 3 we have $a^* = b^*$, which implies $a = b$, as was to be shown.

Theorem 4 will be used to prove univalence on some weaker conditions for two-dimensional cases in Section 7.

4.3 We remark that Theorems 3 and 4, although stated for closed rectangular regions, are immediately true also in arbitrary rectangular regions, either open or semi-closed, which are obtained from closed rectangular regions by replacing either some or all of the defining inequalities by strict inequalities. For if the assertions fail to be true on some rectangular regions, they also fail to hold in some suitable closed rectangular subregions.

4.4 Monotonicity of the Inverse of a Leontief Mapping. A subclass of P-matrices is formed by matrices whose off-diagonal entries are nonpositive. They are often referred to as "of the Leontief type" on account of their prominent role in studies initiated by Leontief in economics. It is very well-known (cf. [2, 3]) that for a matrix of the Leontief type A the following four conditions are equivalent:

(I) There is a vector $x \geqq 0$ such that $Ax > 0$.

(II) A is non-singular and all the entries of A^{-1} are nonnegative.

(III) A is a P-matrix.

(IV) All the upper left-hand corner n principal minors are positive.

Thus the following theorem might be regarded as a generalization of this result on Leontief matrices to nonlinear mappings.

Theorem 5. *Let $F : \Omega \to R^n$, where Ω is a region of R^n, be a differentiable mapping whose Jacobian matrix is of the Leontief type. Then we have*

(i) If Ω is rectangular and the Jacobian matrix is a P-matrix, F is univalent in Ω and F^{-1} is monotonic increasing, that is, $F(a) \leqq F(b)$ for $a, b \in \Omega$ implies $a \leqq b$.

(ii) Suppose that F is univalent in Ω which is an arbitrary open region, not necessarily rectangular. If the inverse F^{-1} is differentiable and monotonic increasing, then the Jacobian matrix of F is a P-matrix.

Proof. *(i)* Univalence is already proved in Theorem 4 together with the preceding remarks. Also, monotonicity is obvious if $n = 1$. Thus, we proceed by induction on n. In general, if $F(a) \leqq F(b)$ for $a = (\alpha_i)$, $b = (\beta_i) \in \Omega$, then Theorem 3 implies that for some k we have $\alpha_k \leqq \beta_k$. Without loss of generality, this k may be assumed to be 1. As was noted, differentiability implies partial differentiability whenever partial differentiation is performable. We may also note that it can be carried out throughout a rectangular region, if one has one-sided partial differentiation in mind at a boundary point. Then we have for $i > 1$

$$f_i(\beta_1, \alpha_2, \ldots, \alpha_n) \leqq f_i(\alpha_1, \alpha_2, \ldots, \alpha_n) \leqq f_i(\beta_1, \beta_2, \ldots, \beta_n)\,, \tag{7}$$

the first inequality holding because $\partial f_i/\partial x_1 \leqq 0$ $(i > 1)$. We now define $\hat{F} : \hat{\Omega} \to R^{n-1}$ by the rule

$$\hat{F}(x_2, \ldots, x_n) = (f_2(\beta_1, x_2, \ldots, x_n), \ldots, f_n(\beta_1, x_2, \ldots, x_n))\,,$$

where $\hat{\Omega}$ is the image of Ω under the projection $(x_1, x_2, \ldots, x_n) \to (x_2, \ldots, x_n)$. Now $\hat{F}(\alpha_2, \ldots, \alpha_n) \leqq \hat{F}(\beta_2, \ldots, \beta_n)$ from (7), and the Jacobian matrix of $\hat{F}$ is again a P-matrix of the Leontief type. Hence by the induction hypothesis $\alpha_i \leqq \beta_i$ $(i = 2, \ldots, n)$ and thus $a \leqq b$, as was to be shown.

(ii) This F turns out to be a topological mapping, and, by the invariance of regionality, $F(\Omega)$ is also an open region. Let a be any point of Ω, and let $a^* = F(a)$. Take some positive vector $u^* > 0$. Then, since $F(\Omega)$ is open, there is some $\varepsilon > 0$ such that $x^*(\lambda) = a^* + \lambda u^* \in F(\Omega)$ for λ with $|\lambda| < \varepsilon$. Let $F^{-1}(x^*(\lambda)) = x(\lambda)$. Then, since F^{-1} is differentiable and monotonic increasing, $x(\lambda)$ is differentiable and $\frac{d}{d\lambda} x(\lambda) = u(\lambda) \geqq 0$ $(|\lambda| < \varepsilon)$. Hence, again differentiating $x^*(\lambda) = F(x(\lambda))$ at $\lambda = 0$, we have $J(a)\, u(o) = u^* > 0$, $u(o) \geqq 0$, where $J(a)$ is the Jacobian matrix of F. Since $J(a)$ is of the Leontief type, in view of the equivalence of (I) and (III), $J(a)$ must be a P-matrix.

5. Case of the Quasi-definite Jacobian Matrix

5.1 Quasi-definite Jacobians. In this section the univalence assertion will be examined for a special P-Jacobian matrix case, namely the quasi-definite Jacobian matrix case. It was noted in Section 2 that positive quasi-definite matrices are P-matrices. Yet separate consideration will be made of this case on account of the weaker condition on the structure of regions than in

Theorem 4. We also note that the result can be applied to a Theorem due to NOSHIRO in the theory of univalent (schlicht) functions.

Theorem 6. *If a differentiable mapping $F : \Omega \to R^n$, where Ω is a convex region (either closed or nonclosed) of R^n, has the Jacobian matrix which is everywhere positive (negative) quasi-definite in Ω, then F is univalent in Ω.*

Proof. Let a, $x \in \Omega$ and $x \neq a$. Let further $x(t) = a + th$, where $x - a = h = (h_i)$. By convexity we have $x(t) \in \Omega (0 \leqq t \leqq 1)$. Define the function $\Phi(t)$ by

$$\Phi(t) = \sum_{i=1}^{n} h_i\big(f_i(x(t)) - f_i(a)\big), \qquad (0 \leqq t \leqq 1)\,.$$

Then, upon differentiation and appealing to the quasi-definiteness, it is immediate to see that

$$\Phi'(t) = \sum_{i,j=1}^{n} f_{ij}(x(t))\, h_i h_j$$

is identically positive or identically negative. Since $\Phi(o) = 0$, $\Phi(1)$ cannot vanish and hence $f_i(x) - f_i(a) \neq 0$ for some i. Therefore, multivalence is ruled out.

5.2 Application to Univalent Functions. The following sufficiency condition for the univalence of an analytic function was first given by NOSHIRO [4]. It is noted that his result can easily be deduced from Theorem 6.

Corollary (NOSHIRO [4]). *Let $f(z)$ be an analytic (complex) function of a complex variable z in a convex region Ω. Then, $f(z)$ is univalent in Ω, if the range of the derivative f' lies in a half-plane not containing the origin in its interior.*

Proof. We first show that it suffices to consider the case in which the real part of the derivative is positive in Ω. In fact, let $f(z) = u + iv$. Thus $f'(z) = u_x + iv_x$ and, by assumption, there are some real numbers p and q such that $pu_x + qv_x > 0$ in Ω. Then, $h(z) = (p - iq)\, f(z)$ is analytic and $h'(z)$ has the positive real part in Ω.

Now, we may assume that $\mathrm{Re}(f'(z)) = u_x > 0$ in Ω, and examine the Jacobian matrix J of the mapping $F : \Omega \to R^2$, where $F(x, y) = (u(x, y), v(x, y))$. It is positive quasi-definite in Ω, since

$$\frac{J + J'}{2} = \begin{pmatrix} u_x & 0 \\ 0 & u_x \end{pmatrix}, \; u_x > 0$$

by virtue of the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$. Hence univalence is assured.

5.3 Non-univalence in Non-convex Regions. In Theorem 6 the convexity assumption is indispensable. An example is given to show that Theorem 6 is no longer true in a non-convex region. In view of the result in 5.2, it suffices to give a non-univalent analytic function $f(z)$ in a non-convex region, whose derivative has positive real part in Ω.

Such an example is given by considering the Joukowski mapping $f(z) = z + \frac{1}{z}$ $(z \neq 0)$ in suitable regions. Let Ω^* be the common exterior portion of the two circles of radius 1/2 and having their centers at (1/2, 0) and (− 1/2, 0). The real part of the derivative $f'(z) = 1 - \frac{1}{z^2}$ may be arranged as

$$\operatorname{Re}(f'(z)) = \frac{\left(\left|z - \frac{1}{2}\right|^2 - \frac{1}{4}\right)\left(\left|z + \frac{1}{2}\right|^2 - \frac{1}{4}\right) + (\operatorname{Im}(z))^2}{|z|^4},$$

which is positive in Ω^*. Yet $f(i) = f(-i) = 0$ while $i, -i \in \Omega^*$. Also one can easily construct a simply-connected subregion of Ω^* which contains i and $-i$.

6. Weakening of the Jacobian Conditions

6.1 Weaker Conditions on Matrices. The purpose of this section is to relax the conditions on the Jacobian in Theorems 4 and 6 to some extent without losing univalence. The proof will be worked out in a straightforward way by means of the Kronecker index. It should be noted, however, that the results will be stated only for open regions in order to avoid complication which might occur on the boundary.

We begin by giving some definitions. An $n \times n$ matrix A is said to be a *weak P-matrix*, if $|A| > 0$ and all the other principal submatrices, of order less than n, have nonnegative determinants. Also, A is said to be *weakly positive quasi-definite*, if $|A| > 0$ and $\frac{1}{2}(A + A')$ is positive semi-definite. A weakly negative quasi-definite matrix can be defined likewise.

6.2 Univalence Theorems under Weaker Conditions. We prove

Theorem 4w. *If $F : \Omega \to R^n$, where Ω is an open rectangular region of R^n, is a differentiable mapping such that the Jacobian matrix $J(x)$ is a weak P-matrix for all x in Ω, then F is univalent in Ω.*

Theorem 6w. *If a differentiable mapping $F : \Omega \to R^n$, where Ω is an open convex region of R^n, has the Jacobian matrix which is a weakly positive quasi-definite matrix in Ω, then it is univalent in Ω.*

Proof. The method of proof is exactly the same for the both theorems. In the course of the proof, however, it should be kept in mind that by a subregion we mean a rectangular subregion in Theorem 4w and a convex subregion in Theorem 6w.

Letting a be an arbitrary point of Ω, we must show that $x \in \Omega$ and $F(x) = F(a)$ imply $x = a$. To this end, it suffices to prove that there is no other solution to $F(x) = F(a)$ than $x = a$ in any bounded closed subregion Δ^* of Ω containing a in its interior. Now take any larger bounded, closed subregion Δ^{**} of Ω such that the interior of Δ^{**} includes Δ^*. Since the Jacobian is positive in Ω, every solution of $F(x) = F(a)$ is isolated, so that there are only a finite number of solutions of $F(x) = F(a)$ in Δ^{**}. Then we can choose a suitable open intermediate region Δ whose boundary Γ contains no solutions of $F(x) = F(a)$. Γ is topologically equivalent to an $(n-1)$-sphere, and may be regarded as a basic (orientation-giving) $(n-1)$-cycle, bounding Δ, when suitably triangulated. We denote this cycle by $\tilde{\Gamma}$. Likewise $F(\tilde{\Gamma})$ stands for the image cycle, while $F(\Gamma)$ denotes the image of Γ as a point set.

Now since $F(\Gamma) \not\ni F(a)$, the order of $F(a)$ relative to the cycle $F(\tilde{\Gamma})$ can be defined and is denoted by $\omega(F(\tilde{\Gamma}), F(a))$. By the Kronecker theorem on indices (cf. [1], pp. 457—478), $\omega(F(\tilde{\Gamma}), F(a))$ equals the sum of the indices of all the solutions to $F(x) = F(a)$ in Δ. Moreover, since the Jacobian is assumed to be identically positive, the index of every solution of $F(x) = F(a)$ equals one. Whence, we have

$$(8) \qquad \omega(F(\tilde{\Gamma}), F(a)) = \text{number of solutions of } F(x) = F(a) \text{ in } \Delta\,.$$

Therefore, we have only to show that $\omega(F(\tilde{\Gamma}), F(a)) = 1$.

To this end we first prove that there is no $x \in \Omega$ such that $x \neq a$, $F(x) - F(a) = -\lambda(x - a)$ for some $\lambda > 0$. In fact, suppose that $F(b) - F(a) = -\lambda(b - a)$ for some $b \in \Omega$ and some $\lambda > 0$. Then, by rearrangement, we have

$$(9) \qquad \lambda a + F(a) = \lambda b + F(b)\,.$$

Now let $\overline{F}(x) = \lambda x + F(x)$ and consider the mapping $\overline{F} : \Omega \to R^n$. This is clearly differentiable. Further, the Jacobian matrix J_λ of this new mapping equals $\lambda I + J$, where I is the identity matrix and J is the Jacobian matrix of F. Let K_λ be an arbitrary principal submatrix of J_λ. Then, $K_\lambda = \lambda I_k + K$, where K is a principal submatrix of J and I_k is the identity matrix of the corresponding order k. It is readily seen that

$$(10) \qquad |K_\lambda| = |\lambda I_k + K| = \lambda^k + \varphi(\lambda)\,,$$

where $\varphi(\lambda)$ is a polynomial of degree at most $k - 1$ and whose coefficients equal certain sums of the principal minors of K. Since these determinants are assumed to be nonnegative in Theorem 4w, $\lambda > 0$ implies $|K_\lambda| > 0$ because of (10). On the other hand, if J is weakly positive quasi-definite as in Theorem 6w, clearly $J_\lambda = \lambda I + J$ is positive quasi-definite because $\lambda > 0$. Therefore the Jacobian of the mapping $\overline{F}$ is identically either a P-matrix or a positive quasi-definite matrix in Ω. But (9) gives $\overline{F}(a) = \overline{F}(b)$ and hence we have, by either Theorem 4 or Theorem 6, $a = b$.

Now, since there is no x in Ω such that $x \neq a$, $F(x) - F(a) = -\lambda(x - a)$ for some $\lambda > 0$, the mapping $F : \Gamma \to R^n$ is homotopic to the mapping $G(x) = F(a) + x - a : \Gamma \to R^n$ in $R^n - \{F(a)\}$ by the homotopy

$$(11) \qquad H(x, t) = (1 - t)\, F(x) + t\, G(x) \; (x \in \Gamma, 0 \leqq t \leqq 1)\,.$$

Whence $\omega(F(\tilde{\Gamma}), F(a)) = \omega(G(\tilde{\Gamma}), F(a))$. But, clearly $\omega(G(\tilde{\Gamma}), F(a)) = 1$ so that, in (8), we have $\omega(F(\tilde{\Gamma}), F(a)) = 1$. Therefore, Δ^* contains, a fortiori, no other solution of $F(x) = F(a)$ than $x = a$, q. e. d.

7. Two-dimensional Cases

The results in this section are rather fragmentary, but are presented because of the directions they suggest for further generalizations of the preceding theorems.

Throughout this section $F : \Omega \to R^2$ is a mapping given by $F(x, y) = (f(x,y), g(x, y))$, $(x, y) \in \Omega$, where Ω is a region of R^2.

7.1 One-signed Principal Minors. We shall prove

Theorem 7. *(i) Suppose that Ω is an arbitrary rectangular region either closed or nonclosed and F has continuous partial derivatives. If then no principal minors of the Jacobian vanish, F is univalent.*

(ii) Suppose that Ω is an open rectangular region and F has continuous partial derivatives. If the Jacobian does not vanish and no diagonal entries of the Jacobian matrix change signs, then F is univalent.

Proof. (i) Since the entries of the Jacobian matrix are continuous, each principal minor is everywhere positive or everywhere negative. Multiplying some or all of the equations by -1 if necessary we may assume that the diagonal entries f_x and g_y are positive. If then the Jacobian is also positive, the theorem is a special case of Theorem 4. Now suppose that the Jacobian is negative, whence $f_x g_y - f_y g_x < 0$. Since $f_x g_y > 0$, this implies $f_y g_x > 0$, so that f_y and g_x never vanish and are of the same sign. Thus, because of continuous differentiability, they are either everywhere positive or everywhere negative. Hence, if $f_y > 0$, $g_x > 0$, then the new mapping $\overline{F} : \Omega \to R^2$, where $\overline{F}(x, y) = (g(x, y), f(x, y))$, has a P-Jacobian matrix. If $f_y < 0$, $g_x < 0$, then the mapping $\tilde{F} : \Omega \to R^2$, where $\tilde{F}(x, y) = (-g(x, y), -f(x, y))$ has a P-Jacobian matrix. Therefore, by Theorem 4, these two mappings are univalent, which proves the univalence of the original mapping.

(ii) The proof can be worked out in the same way. It is noted, however, that in (ii) we have the following three cases: (α) the Jacobian matrix of the original mapping is a weak P-matrix. (β) $\overline{F}$ has a P-Jacobian matrix. (γ) $\tilde{F}$ has a P-Jacobian matrix. Hence, by Theorem 4w, univalence obtains.

7.2 One-signed Entries in a Row. Instead of the one-signedness of diagonal entries, that of the entries of some row of the Jacobian suffices to insure univalence. The results in this line are stated in

Theorem 8. *(i) Suppose that Ω is an open rectangular region and F is differentiable. Assume further that the Jacobian neither vanishes nor changes sign. If there are some real numbers p and q, not all of them zero, such that $pf_x + qg_x$ and $pf_y + qg_y$ do not change sign in Ω, then F is univalent.*

(ii) Suppose that Ω is an arbitrary rectangular region in which F has continuous partial derivatives and the Jacobian never vanishes. If there are some real numbers p and q such that $pf_x + qg_x$ and $pf_y + qg_y$ do not change sign and one of them does not vanish, then F is univalent.

Proof. (i) By simple transformations on the equations and/or variables, we may assume that the Jacobian is positive and $pf_x + qg_x \geqq 0$, $pf_y + qg_y \geqq 0$ in Ω.

In view of the proof of Theorems 4w and 6w in Section 6, it suffices to show that the equation $F(x) = F(a)$ has only the solution $x = a$ in any open bounded rectangular subregion Δ which contains a, but whose boundary Γ contains no solution of the equation. Also, formula (8) is valid. But, this time, the order $\omega(F(\Gamma), F(a))$ equals the winding number, i.e. the number of times $F(x, y)$ goes around $F(a)$, when (x, y) moves counter-clockwise once along Γ. In what follows $F(a)$ may be assumed to be the origin of R^2.

Next we shall show that the winding number here is just one. To this end, consider the function $H(x, y) = pf(x, y) + qg(x, y)$. There are some points of $F(\Gamma)$ on each of the two rays l_1 and l_2 issuing from the origin whose union is the straight line $pf + qg = 0$ in the (f, g)-space. For, otherwise, the winding number would equal zero, contradicting the existence of at least one solution $x = a$ to $F(x) = 0$ in Δ. Now, since by assumption $H_x \geqq 0$ and $H_y \geqq 0$ in Ω, the value of H never decreases when (x, y) moves rightward and upward along Γ. It also never increases when (x, y) moves leftward and downward along Γ. Further the two open regions, defined by $pf + qg > 0$ and $pf + qg < 0$ respectively, in the (f, g)-space intersect $F(\Gamma)$. For, otherweise, again the winding number would be zero. In the light of the above results together with the connectedness of Γ, we can readily trace the movement of $F(x, y)$ when (x, y) moves counterclockwise once along Γ, starting and ending at a suitable point (λ_1, μ_1), and passing through some (λ_2, μ_2), (λ_3, μ_3) and (λ_4, μ_4) in succession. In fact, $F(x, y)$ moves on the ray l_1 between (λ_1, μ_1) and (λ_2, μ_2), stays in the open region defined by $pf + qg > 0$ after leaving (λ_2, μ_2) until reaching (λ_3, μ_3), moves on the other ray l_2 between (λ_3, μ_3) and (λ_4, μ_4), and stays in the opposite open region defined by $pf + qg < 0$ after leaving (λ_4, μ_4) until returning to (λ_1, μ_1). This implies that $F(x, y)$ goes around the origin by 2π, so that the winding number is one. Hence by formula (8), $F(x) = 0$ has only the solution $x = a$ in Δ, as was to be shown.

(ii) We may assume that $p \neq 0$. Then, the original mapping is univalent if and only if the mapping $\overline{F} : \Omega \to R^2$ is so, where $\overline{F}(x, y) = (pf(x, y) + qg(x, y), g(x, y))$. The Jacobian of the latter is that of F multiplied by p. Hence, by further simple transformations if necessary and in view of continuous differentiability, we may finally assume that the original mapping has the following properties: $f_x > 0$ and $f_y \geqq 0$ in Ω.

Now suppose that F is not univalent, so that $F(a, b) = F(c, d) = (\alpha, \beta)$ for some distinct points $(a, b), (c, d) \in \Omega$. Because $f_x > 0$, clearly $b \neq d$, so that we may let $b < d$. Choose any fixed y satisfying $b \leqq y \leqq d$. Then, $f_y \geqq 0$ and $b \leqq y$ imply $\alpha = f(a, b) \leqq f(a, y)$. And similarly $\alpha = f(c, d) \geqq f(c, y)$. By the continuity of f, therefore, $f(x, y) = \alpha$ for some x between a and c. Moreover, for a given y this x is unique, since $f_x > 0$. Thus we obtain an implicit function $x = \varphi(y)$ such that $f(\varphi(y), y) = \alpha$ globally holds in the closed interval $[b, d]$. Then, because $f(x, y)$ has continuous derivatives, φ is also continuously differentiable and the derivative is given by $\varphi'(y) = -f_y(\varphi(y), y)/f_x(\varphi(y), y)$ in $[b, d]$. Let $G(y) = g(\varphi(y), y)$. Then, G is continuously differentiable and $G' = |J|/f_x$, evaluated for $x = \varphi(y)$, where $|J|$ denotes the Jacobian. Since $|J|$ does not vanish and is continuous, $G'(y)$ is either everywhere positive or everywhere negative, so that $G(y)$ is strictly monotonic in $[b, d]$. This contradicts $G(b) = G(d) = \beta$, and nonunivalence is ruled out.

7.3 Some Remarks. Theorems 7 and 8 include as special cases those in which the Jacobian matrices have nonnegative entries. In these cases the diagonal entries of the Jacobian matrices are one-signed, so that Theorem 7, (ii) can be applied. On the other hand, any pair of nonnegative numbers p, q, unless $p = q = 0$, will serve as the p, q in (i) of Theorem 8. Also, for $p = q = 1$, the function $pf + qg$ fulfills the conditions $pf_x + qg_x > 0$, $pf_y + qg_y > 0$ in Ω, for no columns of the Jacobian matrix equal the zero vector, since they are linearly independent. Therefore (ii) of Theorem 8 is also applicable to these cases.

The method of proof for (ii) in Theorem 8 was suggested by SAMUELSON's idea [5], which, however, can apparently not be used in the general case.

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