segregate in back-cross individuals in accordance with genetic expectation. The results of this and previous investigations show beyond reasonable doubt that the species-specific qualities of the serum proteins are determined by gene action and suggest that the total protein complex of the serum is likewise determined by genes.

The genes that in Pearlneck produce the serum antigens are not the same as those that produce the cellular antigens, and possibly are not on the same chromosomes. These findings corroborate the results of parallel investigations of the sera and cells of back-cross hybrids of other combinations of species of doves and pigeons.

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ON THE MOTION OF VORTICES IN TWO DIMENSIONS—I. EXISTENCE OF THE KIRCHHOFF-ROUTH FUNCTION

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1. *Introduction.*—Kirchhoff[1] was the first to establish the existence of "a stream function giving the motion of vortices" in an *unbounded* region. Later, in 1881, Routh[2] enunciated a theorem regarding the transformation of a function of that nature for the case of a *single* vortex moving in a *bounded* region. No proof was given and the existence of such a stream function was not established.[3] In 1921, Lagally[4] established the existence of this "*Routh's stream function*" and the more general "*Kirchhoff's path function*" for a *simply connected bounded* region. An independent proof for the case of a single vortex in such a region was also given by Masotti[5] in 1931, by using the Green function of the first kind. However, the most general case of the motion of a number of vortices in a *multiply connected*

region is not covered by their work. The present article establishes the existence of the "*Kirchhoff-Routh function*"[6] for the general case. The proof makes use of a generalized Green function and may therefore be regarded as a generalization of Masotti's work. Explicit formula of this function is also given. A more detailed treatment of this work will appear elsewhere.

2. *Statement of the Problem.*—Consider a number of isolated free vortices of strengths κ_i ($i = 1, 2, ..., n$) at the points $P_i(x_i, y_i)$ ($i = 1, 2, ..., n$) in an incompressible fluid moving irrotationally in a region R. This region has a number of internal boundaries C_k ($k = 1, 2, ..., m$) and it is (a) bounded externally by a closed curve C_0, or (b) unbounded from the outside, or (c) limited by curves C_0 extending to infinity.

If the ordinary stream function of fluid motion

$$\psi = \psi(x, y;\ x_1, y_1;\ ...;\ x_n, y_n) \tag{2.1}$$

is known (which is obviously independent of the time t explicitly), the components of velocity of the ith vortex ($i = 1, 2, ..., n$) are given by

$$\frac{dx_i}{dt} = u_i = -\left(\frac{\partial\psi_{(i)}}{\partial y}\right)_{P_i}, \quad \frac{dy_i}{dt} = v_i = \left(\frac{\partial\psi_{(i)}}{\partial x}\right)_{P_i} \tag{2.2}$$

where

$$\psi_{(i)} = \psi - \frac{\kappa_i}{2\pi}\log r_i, \quad r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2}, \tag{2.3}$$

and the suffixes P_i denote that the results of differentiation are to be evaluated at the point P_i. For the case where the region R has no solid boundaries and where steady streaming is absent, Kirchhoff[1] has shown that there exists a function

$$W = W(x_1, y_1;\ x_2, y_2;\ ...;\ x_n, y_n), \tag{2.4}$$

such that the motion of the ith vortex is given by

$$\kappa_i \frac{dx_i}{dt} = \kappa_i u_i = -\frac{\partial W}{\partial y_i}, \quad \kappa_i \frac{dy_i}{dt} = \kappa_i v_i = \frac{\partial W}{\partial x_i}. \tag{2.5}$$

The object of this article is to show that this result of Kirchhoff can be generalized to the motion of vortices in a region R of the general type described above. In another article, we shall derive the law of transformation of the *Kirchhoff-Routh function* W under a conformal transformation.

We shall first define a generalized Green function particularly suited to the study of vortex motion. The essence of this paper, like Masotti's work, lies in the application of the reciprocity property of a properly defined Green function. (Cf. equations (3.5) and (3.6)).

3. *The Green Function.*—Let us define a function $G(x, y;\ x_0, y_0)$ with respect to two points $P(x, y)$ and $P_0(x_0, y_0)$ in the region R by the following three conditions.

(i) The function

$$g(x, y;\ x_0, y_0) = G(x, y;\ x_0, y_0) - \frac{1}{2\pi}\log r_0 \qquad \left(r_0 = \sqrt{(x - x_0)^2 + (y - y_0)^2}\right) \quad (3.1)$$

is harmonic with respect to (x, y) throughout the region R including the point P_0; thus,

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0. \quad (3.2)$$

(ii) If $\partial G/\partial n$ is the normal derivative of the function G on a curve (with x, y as variables), then

$$G = A_k \text{ on } C_k, \quad \int_{C_k'} \frac{\partial G}{\partial n}\, ds = 0 \quad (k = 1, 2, \ldots, m) \quad (3.3)$$

for each of the inner boundaries C_k; where ds is an element of arc of the closed analytical curve C_k', which encloses C_k but neither the other boundary curves nor the point P_0. The positive normal to C_k' is drawn toward C_k and the positive direction of C_k' is taken so that the main part of R (including P_0) lies to its left. Similar positive directions are used for any other curve.

(iii*a*) If the region R has a closed outer boundary C_0, then

$$G(x, y;\ x_0, y_0) = 0 \quad \text{over } C_0. \quad (3.4a)$$

(iii*b*) If the region R extends to infinity in all directions, the function $G(x, y;\ x_0, y_0)$ behaves as follows:[7]

$$\left.\begin{aligned} &G(x, y;\ x_0, y_0) = \frac{1}{2\pi}\log r_0 + O\left(\frac{1}{r_0}\right), \\ &\frac{\partial G}{\partial s} = O\left(\frac{1}{r_0^2}\right), \frac{\partial G}{\partial n} = \frac{1}{2\pi r_0} + O\left(\frac{1}{r_0^2}\right), \end{aligned}\right\} \text{ over a very large circle of radius } r_0. \quad (3.4b)$$

where $\partial G/\partial s$ is the tangential derivative along the circle.

(iii*c*) If the region R has boundaries C_0 extending to infinity, the function $G(x, y;\ x_0, y_0)$ behaves as follows:[7]

$$\left.\begin{aligned} &G(x, y;\ x_0, y_0) = O \quad \text{over } C_0, \\ &G(x, y;\ x_0, y_0) = o(1) \quad \text{over a very large circle of radius } r_0. \end{aligned}\right\} \quad (3.4c)$$

Koebe[8] established the existence and uniqueness of this Green function G by resolving it into a linear combination of two sets of basic functions: (*a*) the Green function of the first kind and (*b*) the set of harmonic functions each of which takes the value unity on a particular one of the boundary curves and the value zero on all the others. With the help of this resolution, the present writer finds it possible to prove the symmetry property of this Green function by using standard methods.[9]

We summarize the results in the following lemma.

LEMMA I. *The function $G(x, y;\ x_0, y_0)$ defined by the conditions (3.2)–(3.4) exists uniquely, and is a generalized Green function satisfying the reciprocity property*

$$G(x, y;\ x_0, y_0) = G(x_0, y_0;\ x, y). \tag{3.5}$$

The reciprocity property immediately leads to the following important result (cf. (3.1)):

$$\left.\begin{aligned} \frac{\partial}{\partial x_0} g(x_0, y_0;\ x_0, y_0) &= 2 \lim_{P \to P_0} \frac{\partial}{\partial x} g(x, y;\ x_0, y_0), \\ \frac{\partial}{\partial y_0} g(x_0, y_0;\ x_0, y_0) &= 2 \lim_{P \to P_0} \frac{\partial}{\partial y} g(x, y;\ x_0, y_0). \end{aligned}\right\} \tag{3.6}$$

4. *Kirchhoff's Equations.*—Let us now apply our function G to the hydrodynamical problem of vortex motion with the stream function (2.1).

The motion given by the stream function

$$\Psi = \kappa_0 G(x, y;\ x_0, y_0) \tag{4.1}$$

may be called the motion *due* to a vortex κ_0 at the point $P_0(x_0, y_0)$. It is a possible potential motion in the region R with the required singularity, and has no circulation around any one of the inner boundaries. Furthermore, if the region R extends to infinity, the flow *across* an arc of the circle r_0 = constant (contributing to outward flux) approaches zero and the flow *along* it (contributing to circulation) is finite (or approaches zero), as r_0 becomes infinite.

If we subtract the stream functions *due* to all the vortices from the complete stream function (2.1), the remainder is a stream function giving a possible potential motion in the region R. Now, this motion is uniquely defined when the circulation around each of the curves C_k and the fluid velocity at infinity are given. Since these conditions are the same for the complete stream function as for this part, this motion is actually independent of the (variable) positional coordinates (x_i, y_i) of the vortices. It depends on the (constant) strengths κ_i, only if the conditions defining it happen to be so. We may therefore call it the motion "*due* to outside agencies."

We summarize our results in the following basic lemma.

LEMMA II. *If n vortices of strengths κ_i $(i = 1, 2, \ldots, n)$ are present in an* *incompressible fluid at the points* $P_i(x_i, y_i)$ $(i = 1, 2, \ldots, n)$ *in a general region bounded by fixed boundaries, the stream function of fluid motion is given by*

$$\psi(x, y;\ x_1, y_1, \ \ldots;\ x_n, y_n) = \psi_0(x, y) + \sum_{i=1}^{n} \kappa_i G(x, y;\ x_i, y_i), \quad (4.2)$$

where the properties of $G(x, y;\ x_i, y_i)$ *are given in Lemma I, and* $\psi_0(x, y)$ *is the stream function of the motion due to outside agencies, independent of* $P_i(x_i, y_i)$ *(and of* κ_i*).*

With this result at hand, we can at once establish the existence of the Kirchhoff-Routh function.

THEOREM I. *For the motion of vortices of strengths* $\kappa_i (i = 1, 2, \ldots, n)$ *in a general region R bounded by fixed boundaries, there exists a Kirchhoff-Routh function* $W(x_1, y_1;\ x_2, y_2;\ \ldots;\ x_n, y_n)$ *such that*

$$\left.\begin{aligned} \kappa_i \frac{dx_i}{dt} &= \kappa_i u_i = -\frac{\partial W}{\partial y_i}, \\ \kappa_i \frac{dy_i}{dt} &= \kappa_i v_i = \frac{\partial W}{\partial x_i}, \end{aligned}\right\} \quad (4.3)$$

where $P_i(x_i, y_i)$ $(i = 1, 2, \ldots, n)$ *are the instantaneous positions of the vortices. The function W is given by*

$$W = \sum_{i=1}^{n} \kappa_i \psi_0(x_i, y_i) + \sum_{\substack{i,j=1 \\ (i>j)}}^{n} \kappa_i \kappa_j\, G\,(x_i, y_i;\ x_j, y_j) + \frac{1}{2} \sum_{i=1}^{n} \kappa_i^2 g(x_i, y;\ x_i, y_i). \quad (4.4)$$

This can be at once verified by comparing the results for u_i and v_i obtained from (4.3), (4.4) and (3.6) with those obtained from (2.2), (2.3) and (4.2). We note that the system of *Kirchhoff's equations* (4.3) is a Hamiltonian system of differential equations in the set of variables $\sqrt{\kappa_i} x_i$ and $\sqrt{\kappa_i} y_i$, $(i = 1, 2, \ldots, n)$.

It is not difficult to show that when the vortices are fixed the *total* force reacting on all the solid boundaries is given by

$$X = \sum_{i=1}^{n} \rho(\partial W/\partial x_i), \ Y = \sum_{i=1}^{n} \rho(\partial W/\partial y_i),$$

ρ being the density of the fluid. It can also be shown that ρW is a *measure* of the kinetic energy of fluid motion, so that equation (4.3) leads to the law of conservation of energy, W = constant.

[1] Kirchhoff, G., *Vorlesungen über mathematische Physik, Mechanik*, p. 255 ff. See also Lamb, H., *Hydrodynamics*, 1932, p. 230.

[2] Routh, E. J., *Proc. Lond. Math. Soc.*, **12**, 83 (1881).

[3] Even in the standard books of recent years, the treatment is far from complete; cf. Lamb, H., *Hydrodynamics*, 1932, pp. 219–236; Ramsey, A. S., *A Treatise of Hydro-*

mechanics, Pt. II, Hydrodynamics, 1935, pp. 219–232; Milne-Thomson, L. M., *Theoretical Hydrodynamics*, 1938, pp. 323–348.

[4] Lagally, M., *Math. Zeits.*, **10**, 231–239 (1921).

[5] Masotti, A., *Atti Pontif. Accad. Sci. Nuovi Lincei*, **84**, 209–216, 235–245, 464–467, 468–473, 623–631 (1931). Also, *Seminario Matematico e Fisico di Milano*, **6**, 3–53 (1932).

[6] The function called by Lagally the Kirchhoff's path function shall be called in this paper the "Kirchhoff-Routh function." The study of the function called by him the Routh's stream function is not of much importance, because it is merely a special application of the other (cf. equation (6.1)).

[7] For the definition of $O(\ )$ and $o(\ )$, cf. Titchmarsh, E. C., *Theory of Functions*, 1932, p. 1, Oxford.

[8] Koebe, P., *Acta Math.*, **41**, 306–344 (1918). Note that our function in case (*b*) is the function with two singularities, one at P_0, the other at infinity.

[9] Cf. Kellogg, O. D., *Foundations of Potential Theory*, 1929, pp. 238–240, Berlin. Note that no assumption is made regarding the nature of the boundaries $C_0, C_1, \ldots, C_k$.

ON THE MOTION OF VORTICES IN TWO DIMENSIONS—II SOME FURTHER INVESTIGATIONS ON THE KIRCHHOFF-ROUTH FUNCTION

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5. *Conformal Transformation.*—We shall now investigate the behavior of the Kirchhoff-Routh function (whose existence we have established in the preceding article) under a conformal transformation of fluid motion.

THEOREM II (*Generalized Routh's theorem*).—*Under a conformal transformation*

$$\tilde{z} = f(z) \tag{5.1}$$

which derives the motion in the $\tilde{z}$-plane from that in the z-plane, the Kirchhoff-Routh function for the new motion is given by

$$\tilde{W} = W + \sum_{i=1}^{n} \frac{\kappa_i^2}{4\pi} \log \left|\frac{dz}{d\tilde{z}}\right|_{P_i} \tag{5.2}$$

Proof. If $F(z)$ is the complex stream function in the z-plane, we have (*cf.*(2.3))

$$-u_i + iv_i = \lim_{P \to P_i} \frac{d}{dz}\left\{F(z) - \frac{i\kappa_i}{2\pi}\log(z - z_i)\right\}. \tag{5.3}$$

We mark every quantity in the $\tilde{z}$-plane with a curl. The complex stream function for the new motion is then