Rayleigh–Bénard Convection – Linear Theory

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Overview

You have already studied parallel flow instability from the point of view of linear theory. There, the governing equation is the Orr–Sommerfeld equation, the eigenvalues of which determine the stability of the parallel flow to small-amplitude disturbances. In general, there are no closed-form solutions to the Orr–Sommerfeld equation, either for the eigenfunctions or the eigenvalues. Therefore, the aim in these lectures is to look at a non-trivial but highly relevant physical system where the linear theory admits analytical expressions. This is the case of Rayleigh–Bénard instability.

The idea behind the Rayleigh–Bénard instability is to take a uniform homogeneous fluid sandwiched between two plates, and to heat the bottom plate so that a density gradient emerges, with a cooler, denser layer lying on top of a hotter, less dense layer, thereby inducing an unstable stratification. Beyond a threshold values, this configuration becomes unstable, triggering a convective motion that counteracts the unstable stratification. The mathematics of this flow instability is introduced in these lectures.

1 Governing equations

We start with the governing Navier–Stokes equations of incompressible flow in an arbitrary domain:

$$\rho\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)=-\frac{\partial p}{\partial x_i}+\mu\nabla^2 u_i+\rho g_i,$$

where all the symbols have their usual meaning and the gravity vector is $(g_1,g_2,g_3)=(0,0,-g)$, such that gravity points in the negative z-direction. The Navier–Stokes equation

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is supplemented with the incompressibility condition

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0.$$

To close the Navier–Stokes equations, further conditions (in addition to the incompressibility relation) are required. In particular, it is necessary to prescribe the behaviour of the density function. In the present application, we are interested in fluid behaviour in the presence of a temperature gradient, so it is sensible to focus on a model where the density depends on temperature (T), wherein the simplest possible model is a linear relation:

$$\rho = \rho_0 + \delta\rho, \qquad \delta\rho = -\rho_0 \alpha (T - T_0),$$

where ρ_0 is the reference density, $\delta\rho$ is a fluctuation which depends linearly on temperature. Also, T_0 is the reference temperature, with $T = T_0 \iff \rho = \rho_0$. Finally, the quantity $\alpha > 0$ is the coefficient of volume expansion. We are not done yet: the evolution of temperature field $T(\boldsymbol{x}, t)$ must be precribed. However, this can be accurately modelled by an advection-diffusion equation:

$$\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \kappa \nabla^2 T,$$

where $\kappa > 0$ is the thermal diffusivity. We assemble all our equations into a single mathematical model:

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i + \rho g_i, \tag{1a}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0, \tag{1b}$$

$$\rho = \rho_0 + \delta\rho, \qquad \delta\rho = -\rho_0 \alpha (T - T_0), \tag{1c}$$

$$\frac{\partial T}{\partial t} + u_i \frac{\partial T}{\partial x_i} = \kappa \nabla^2 T, \tag{1d}$$

In practice, the density variations are quite small, and an approximation can be made wherein **density variations are considered only in the buoyancy (gravity) term**. This is called the **Boussinesq Approximation** (for a full justification of this approximation, see pages 16-17 in [1]. Thus, Equations (1a)–(1b) simplify to

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + \left(1 + \frac{\delta\rho}{\rho_0} g_i\right), \qquad \nu = \mu/\rho_0, \tag{2a}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2b}$$

while the density and temperature laws remain unchanged. This is a great simplification, as the density in the Navier–Stokes equations is now 'almost' a constant.

2 The base state

We study a time-independent base state involving no flow, with $u_i = 0$ and a static temperature distribution, such that

$$\nabla^2 T = 0.$$

We also focus on a two-dimensional geometry for now, in the (x, z) plane, such that the solution of the Laplace equation for temperature reads

$$T = T_0 + Ax + Bz,$$

where A and B are constants. However, we specialize without loss of generality to a situation where the temperature gradient is imposed in the z-direction only, such that $A = 0$. Also, we focus on the more interesting case of an **adverse** temperature gradient, such that $B = -\beta$, with $\beta > 0$, and such that

$$T = T_0 - \beta z.$$

Thus, compared to a baseline at $z = 0$ where the temperature is T_0, high up where $z > 0$ it is relatively colder and low down where $z < 0$ it is relatively hotter. Next, using $\rho = \rho_0 + \delta\rho = \rho_0 - \alpha\rho_0(T - T_0)$ we obtain

$$\rho = \rho_0(1 + \alpha\beta z).$$

Again, compared to a baseline at $z = 0$ where the density is ρ_0, high up where $z > 0$ the fluid is both relatively cool and relatively **more dense** while low down where $z < 0$ it is relatively hot and less dense. This is the notion of an adverse temperature gradient - the temperature and density gradients are going in opposite directions. The last part of the characterization of the base state is the determination of the pressure. We have the w-velocity equation:

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} - \rho_0 g(1 + \alpha\beta z) + \nu\nabla^2 w.$$

With $w = 0$ this gives

$$\frac{\partial p}{\partial z} = -\rho_0 g(1 + \alpha\beta z). \tag{3}$$

The analogous u-velocity equation gives $\partial p/\partial x = 0$. Note that Equation (3) is the equation of **hydrostatic balance**: the pressure drop and the gravity force are balanced. Solving Equation (3) gives

$$p = -\rho_0 g\left(z + \tfrac{1}{2}\alpha\beta z^2\right).$$

We now characterize the base state in full by assembling our results in one place:

$$u_i = 0, \qquad p = -\rho_0 g\left(z + \tfrac{1}{2}\alpha\beta z^2\right). \tag{4a}$$

$$T = T_0 - \beta z, \qquad \rho = \rho_0(1 + \alpha\beta z). \tag{4b}$$

3 Linear stability analysis

Equations (4) are the time-independent **base state** of the problem. This solution would appear to be unstable as the stratification is (apparently) itself unstable: a denser fluid sits on top of a less dense fluid. The idea of the remainder of this chapter is to investigate this stability problem. We do so by introducing perturbations:

$$u_i = \underbrace{0}_{\text{base state}} + \underbrace{u_i}_{\text{perturbations}}$$

and

$$T' = \underbrace{T_0 - \beta z}_{\text{base state}} + \underbrace{\theta}_{\text{perturbations}}.$$

We assume that the perturbations are **small** in the sense that the equations of motion for (u_i, T') can be **linearized** without any loss of accuracy in the modeling. The linearized equations of motion read

$$\frac{\partial u_i}{\partial t} = -\frac{\partial}{\partial x_i}\left(\frac{\delta p}{\rho_0}\right) + \delta_{i,z} g\alpha\theta + \nu\nabla^2 u_i, \quad \text{(5a)}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \quad \text{(5b)}$$

$$\frac{\partial \theta}{\partial t} = w\beta + \kappa\nabla^2\theta. \quad \text{(5c)}$$

Here, δp is the perturbation pressure.

Exercise 1. *Prove Equation* (5) *by carrying out the relevant linearization.*

In incompressible flow wherein the density is a fixed constant, the pressure is always a 'bad' variable because it does not have its own equation (it is determined implicitly via the relation $\partial_i u_i = 0$). Thus, we always try to eliminate the pressure from the equations of motion. We do that here by considering again the momentum equations:

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{\delta p}{\rho_0}\right) + \nu\nabla^2 u, \quad (6)$$

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial z}\left(\frac{\delta p}{\rho_0}\right) + g\alpha\theta + \nu\nabla^2 w, \quad (7)$$

and by taking $\partial_z(6) - \partial_x(7)$; the result is

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = -g\alpha\frac{\partial \theta}{\partial x} + \nu\nabla^2\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right). \quad (8)$$

We take ∂_x(8) and obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x \partial z}-\frac{\partial^2 w}{\partial x^2}\right)=-g\alpha\frac{\partial^2\theta}{\partial x^2}+\nu\nabla^2\left(\frac{\partial^2 u}{\partial x\partial z}-\frac{\partial w}{\partial x^2}\right). \tag{9}$$

We use the incompressibility condition $\partial_x u + \partial_z w$ to write

$$\frac{\partial^2 u}{\partial x \partial z}=-\frac{\partial^2 w}{\partial z^2}.$$

Hence, Equation (9) becomes

$$\frac{\partial}{\partial t}\left(-\frac{\partial^2 w}{\partial z^2}-\frac{\partial^2 w}{\partial x^2}\right)=-g\alpha\frac{\partial^2\theta}{\partial x^2}+\nu\nabla^2\left(-\frac{\partial^2 w}{\partial z^2}-\frac{\partial w}{\partial x^2}\right). \tag{10}$$

Finally then, we obtain

$$\frac{\partial}{\partial t}\nabla^2 w=+g\alpha\frac{\partial^2\theta}{\partial x^2}+\nu\nabla^4 w.$$

We now assemble here in one place the two closed stability equations for the perturbation velocity and temperature:

$$\frac{\partial}{\partial t}\nabla^2 w=+g\alpha\frac{\partial^2\theta}{\partial x^2}+\nu\nabla^4 w, \tag{11a}$$

$$\frac{\partial\theta}{\partial t}=w\beta+\kappa\nabla^2\theta \tag{11b}$$

At this point, it is appropriate to discuss boundary conditions. We assume that the flow is unbounded in the x-direction, with $-\infty < x < \infty$, and that the flow is confined in the z-direction by two parallel plates, located at $z = 0$ and $z = d$. The temperature is maintained at fixed values at the plate walls, such that the temperature perturbations vanish at those walls:

$$\theta = 0, \qquad z = 0, d.$$

Also, because of the no-flux/no penetration conditions at the walls, we have

$$w = 0, \qquad z = 0, d.$$

Now, the equation to solve is fourth-order in w, so further boundary conditions are required. Because of no-slip, we have $u = 0$ on the walls, hence $\partial u/\partial x = 0$ on the walls. By continuity, this means that

$$\frac{\partial w}{\partial z} = 0, \qquad z = 0, d,$$

and this gives the required number of boundary conditions necessary to solve Equations (11).

4 Normal-mode solution

Because of the translational invariance of the equations (11) in the x-direction, it makes sense to introduce a trial solution $w \propto \mathrm{e}^{\mathrm{i}kx}$ and $\theta \propto \mathrm{e}^{\mathrm{i}kx}$ representing a plane wave, where k is the wavenumber. Indeed, it also makes sense to introduce exponential time dependence (in a standard way) such that the following **normal-mode solution** is proposed:

$$w = \mathrm{e}^{pt+\mathrm{i}kx}W(z), \quad (12a)$$
$$\theta = \mathrm{e}^{pt+\mathrm{i}kx}\Theta(z). \quad (12b)$$

Substitution of Equations (12) into Equations (11) yields

$$p\left(\partial_z^2 - k^2\right)W = \nu\left(\partial_z^2 - k^2\right)^2 W - g\alpha k^2\Theta,$$
$$p\Theta = \beta W + \kappa\left(\partial_z^2 - k^2\right)\Theta.$$

Before going any further, we reduce the number of parameters in these equations by rescaling as follows:

$$\left(\partial_z^2 - k^2 - \frac{p}{\kappa}\right)\Theta = -\frac{\beta}{\kappa}W, \quad (13a)$$
$$\left(\partial_z^2 - k^2\right)\left(\partial_z^2 - k^2 - \frac{p}{\nu}\right)W = +\frac{g\alpha}{\nu}k^2\Theta. \quad (13b)$$

We introduce a non-dimensional z-coordinate $\widetilde{z} = z/d$, with

$$\frac{d}{dz} = \frac{d\widetilde{z}}{dz}\frac{d}{d\widetilde{z}} = \frac{1}{d}\frac{d}{d\widetilde{z}},$$

and the equations (13) become

$$\left(\partial_{\widetilde{z}}^2 - d^2k^2 - \frac{pd^2}{\nu}\frac{\nu}{\kappa}\right)\Theta = -\frac{\beta d^2}{\kappa}W, \quad (14a)$$
$$\left(\partial_{\widetilde{z}}^2 - d^2k^2\right)\left(\partial_{\widetilde{z}}^2 - d^2k^2 - \frac{pd^2}{\nu}\right)W = +\frac{g\alpha d^2}{\nu}(d^2k^2)\Theta. \quad (14b)$$

We identify

$$\mathrm{Pr} = \frac{\nu}{\kappa}, \qquad \sigma = \frac{pd^2}{\nu}, \qquad [\sigma] = 1,$$

where $\mathrm{Pr} = \nu/\kappa$ is the **Prandtl number**. Thus, Equations (14) become

$$\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2 - \sigma\mathrm{Pr}\right)\Theta = -\frac{\beta d^2}{\kappa}W, \quad (15a)$$
$$\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2\right)\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2 - \sigma\right)W = +\frac{g\alpha d^2}{\nu}\widetilde{k}^2\Theta, \quad (15b)$$

where $\widetilde{k} = dk$ is a dimensionless wavenumber . We combine the Θ and W-equations by taking the W-equation and operating on it with $(\partial_z^2 - k^2 - \sigma\,\mathrm{Pr})$. We obtain

$$\begin{aligned}
\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2 - \sigma\mathrm{Pr}\right)\left[\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2\right)\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2 - \sigma\right)W\right] &= \left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2 - \sigma\mathrm{Pr}\right)\left[\frac{g\alpha d^2}{\nu}\widetilde{k}^2\Theta\right], \\
&= \frac{g\alpha d^2}{\nu}\widetilde{k}^2\left[\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2 - \sigma\mathrm{Pr}\right)\Theta\right], \\
&= \frac{g\alpha d^2}{\nu}\widetilde{k}^2\left[-\frac{\beta d^2}{\kappa}W\right], \\
&= -\frac{g\alpha\beta d^4}{\nu\kappa}\widetilde{k}^2 W.
\end{aligned}$$

We introduce

$$\mathrm{Ra} = \frac{g\alpha\beta d^4}{\nu\kappa}$$

as the **Rayleigh number** and we have the following single stability equation:

$$\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2 - \sigma\mathrm{Pr}\right)\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2 - \sigma\right)\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2\right) W = -\mathrm{Ra}\,\widetilde{k}^2 W. \tag{16a}$$

Exercise 2. *Show that the Rayleigh number is dimensionless.*

Viewing the eigenvalue problem as an equation in the single variable W, it can be noted that the ordinary differential equation to solve is sixth order. We already have the boundary conditions

$$W = W' = 0, \qquad z = 0, 1, \tag{16b}$$

giving four boundary conditions. We need two more boundary conditions to close the problem. However, since $(\partial_z^2 - k^2)(\partial_z^2 - k^2 - \sigma)W = (g\alpha d^2/\nu)k^2\Theta$, and since $\Theta = 0$ on the boundaries, the remaining two boundary conditions are given by

$$\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2 - \sigma\right)\left(\partial_{\widetilde{z}}^2 - \widetilde{k}^2\right) W = 0, \qquad z = 0, 1. \tag{16c}$$

Thus, we have an ordinary differential equation in the eigenvalue σ. Before attempting various approaches to solve for σ as a function of k explicitly, we first of all investigate the properties of this equation using *a priori* methods. Following standard practice, in the remainder of this Chapter we omit the tildes over the dimensionless variables.

Remark 1. *A normal-mode trial solution is possible when a problem possesses translational symmetry.*

For, consider a generic linear problem

$$\frac{\partial\phi}{\partial t} = \mathcal{L}\phi,$$

where $\phi = \phi(x, t)$ is some scalar field and $\mathcal{L}$ is a linear operator depending on ∂_x and higher derivatives, such that $\mathcal{L}$ is translation invariant:

$$\mathcal{L}(x) = \mathcal{L}(x + a), \qquad \text{for all a} \in \mathbb{R}.$$

Introduce the translation operator $\mathcal{T}$:

$$\mathcal{T}_a\phi(x) = \phi(x + a).$$

Thus, $\mathcal{T}\mathcal{L}\phi = \mathcal{L}\mathcal{T}\phi$, since $\mathcal{T}$ has no effect on $\mathcal{L}$. In other words, $\mathcal{T}$ and $\mathcal{L}$ commute as operators. There is a theorem in Linear Algebra that says that if two operators commute, then they share the same eigenvectors (eigenfunctions). And, since $\mathrm{e}^{\mathrm{i}kx}$ is an eigenfunction of $\mathcal{T}$, it must also be an eigenfunction of $\mathcal{L}$.

In more detail, $\mathrm{e}^{\mathrm{i}kx}$ is an eigenfunction of $\mathcal{T}$ because

$$\mathcal{T}\mathrm{e}^{\mathrm{i}kx} = \mathrm{e}^{\mathrm{i}k(x+a)} = \mathrm{e}^{\mathrm{i}ka}\mathrm{e}^{\mathrm{i}kx},$$

hence $\mathrm{e}^{\mathrm{i}kx}$ is the eigenfunction with eigenvalue $\mathrm{e}^{\mathrm{i}ka}$. ∎

5 A priori methods for the stability equation

Remark 2. *The proof of the theorem was skipped in class.*

We prove the following theorem:

Theorem. *Consider the eigenvalue problem given by Equation* (16)*. The eigenvalue σ is purely real and therefore, the transition from stability to instability is given by $\sigma = 0$.*

Proof: Introduce

$$G = (\partial_z^2 - k^2)W, \qquad F = (\partial_z^2 - k^2)(\partial_z^2 - k^2 - \sigma)W,$$

hence $F = (\partial_z^2 - k^2 - \sigma)G$. The boundary conditions in Equation (16) imply that $F = 0$ at $z = 0$ and $z = 1$. Also, the eigenvalue equation can be rewritten as

$$(\partial_z^2 - k^2 - \mathrm{Pr}\,\sigma)F = -\mathrm{Ra}\,k^2 W.$$

We multiply both sides of this equation by F^* and integrate from $z = 0$ to $z = 1$. Now,

$$\int_0^1 F^*\partial_z^2 F\,\mathrm{d}z = -\int_0^1 |\partial_z F|^2\,\mathrm{d}z,$$

in view of the boundary conditions on F at $z = 0$ and $z = 1$. Thus, we obtain

$$\int_0^1 \left[|\partial_z F|^2 + (k^2 + \mathrm{Pr}\,\sigma)|F|^2\right]\mathrm{d}z = \mathrm{Ra}\int_0^1 F^* W\,\mathrm{d}z.$$

Now consider

$$\begin{aligned}
\int_0^1 F^* W\,\mathrm{d}z &= \int_0^1 W(\partial_z^2 - k^2 - \sigma^*)G^*\,\mathrm{d}z, \\
&= \int_0^1 W\partial_z^2 G^*\,\mathrm{d}z - (k^2 + \sigma^*)\int_0^1 WG^*\,\mathrm{d}z, \\
&= \int_0^1 G^*\partial_z^2 W\,\mathrm{d}z - (k^2 + \sigma^*)\int_0^1 WG^*\,\mathrm{d}z, \\
&= \int_0^1 G^*\left[(\partial_z^2 - k^2) - \sigma^*\right]W\,\mathrm{d}z,
\end{aligned}$$

where we have used integration by parts repeatedly to show that

$$\int_0^1 W\partial_z^2 G^*\,\mathrm{d}z = \int_0^1 G^*\partial_z W\,\mathrm{d}z,$$

using further the fact that $W = \partial_z W = 0$ on $z = 0, 1$. Now,

$$\begin{aligned}
\int_0^1 F^* W\,\mathrm{d}z &= \int_0^1 G^*\left[(\partial_z^2 - k^2) - \sigma^*\right] W\,\mathrm{d}z, \\
&= \int_0^1 G^*\left(G - \sigma^* W\right)\mathrm{d}z, \\
&= \int_0^1 |G|^2\,\mathrm{d}z - \sigma^* \int_0^1 G^* W, \\
&= \|G\|_2^2 - \sigma^* \int_0^1 \left[\left(\partial_z^2 - k^2\right) W^*\right] W\,\mathrm{d}z, \\
&= \|G\|_2^2 + \sigma^* \int_0^1 \left(|\partial_z W|^2 + k^2|W|^2\right)\mathrm{d}z
\end{aligned}$$

Putting it all together, we have

$$\|\partial_z F\|_2^2 + (k^2 + \mathrm{Pr}\,\sigma)\|F\|_2^2 = \mathrm{Ra}\left[\|G\|_2^2 + \sigma^*\left(\|\partial_z W\|_2^2 + k^2\|W\|_2^2\right)\right]$$

Take imaginary parts on both sides of this equation:

$$\mathrm{Pr}\,\Im(\sigma)\|F\|_2^2 = -\mathrm{Ra}\Im(\sigma)\left(\|\partial_z W\|_2^2 + k^2\|W\|_2^2\right),$$

where the mysterious minus sign emerges on the right-hand side because it is σ^* that appears there, not σ. Hence,

$$\Im(\sigma)\left[\mathrm{Pr}\|F\|_2^2 + \mathrm{Ra}\left(\|\partial_z W\|_2^2 + k^2\|W\|_2^2\right)\right] = 0.$$

Now, the quantity inside the square brackets is positive definite, so we are forced to conclude that

$$\Im(\sigma) = 0.$$

For the second part of the theorem, we start with the fact that the system changes from stable to unstable when

$$\Re(\sigma) = 0.$$

However, σ is purely real, so this condition for a change in the stability amounts to

$$\sigma = 0.$$

This concludes the proof. ▮

In stability theory it is of interest to construct the **neutral curve** $\Re(\sigma) = 0$ as a function of the problem parameters (in this case $(\mathrm{Ra}, \mathrm{Pr}, k)$). For the Rayleigh–Benard convection, we have shown that the neutral curve amounts to $\sigma(\mathrm{Ra}, \mathrm{Pr}, k) = 0$. In the next section we find semi-explicit solutions for this neutral curve.

Remark 3. *Quite generally, in linear stability theory, the condition*

$$\Re(\sigma) = 0$$

is called the **threshold** *or the point of criticality. This is where the system switches between stable and unstable states. In the case of Rayleigh–Bénard convection, the imaginary part of* σ *is always zero, so for this one instance only, the point of criticality is simply* $\sigma = 0$. *This simplifies the next piece of analysis.*

6 Explicit solution for the neutral curve

From the previous section, it is known that the neutral curve occurs when $\sigma = 0$. Therefore, we set $\sigma = 0$ in the eigenvalue problem (16) to obtain the following simplified equations:

$$\left(\partial_z^2 - k^2\right)^3 W = -\mathrm{Ra}\, k^2 W.$$

Instead of placing the plates at $z = 0, 1$, we instead set up the problem in a more symmetric manner, such that $z \in (-1/2, 1/2)$. Thus, the relevant boundary conditions are imposed as follows:

$$W = W' = (\partial_z^2 - k^2)^2 W = 0 \qquad \text{on } z = \pm\tfrac{1}{2}.$$

We immediately make a trial solution

$$W = \mathrm{e}^{\pm qz}$$

such that

$$(q^2 - k^2)^3 = -\mathrm{Ra}\, k^2.$$

We call

$$\mathrm{Ra}\, k^2 = \tau^3 k^6$$

hence

$$(q^2 - k^2)^3 = -\tau^3 k^6,$$

and

$$q^2 = k^2 + (-1)^{1/3} \tau k^2.$$

Note also:

$$\tau = (\mathrm{Ra}/k^4)^{1/3}.$$

The three cube roots of unity are

$$(-1)^{1/3} = -1, \qquad \tfrac{1}{2}\left(1 \pm \mathrm{i}\sqrt{3}\right),$$

hence

$$q^2 = -k^2(\tau - 1), \qquad q^2 = k^2 \left[1 + \tau \tfrac{1}{2}\left(1 \pm \mathrm{i}\sqrt{3}\right)\right].$$

Taking square roots, we obtain the following six square-root solutions:

$$\pm \mathrm{i} q_0, \qquad \pm q, \qquad \pm q^*,$$

where $q_0 = k\sqrt{\tau - 1}$ and

$$\Re(q) \;:=\; q_1 = k\left[\tfrac{1}{2}\sqrt{1+\tau+\tau^2} + \tfrac{1}{2}\left(1 + \tfrac{1}{2}\tau\right)\right]^{1/2}, \tag{17a}$$

$$\Im(q) \;:=\; q_2 = k\left[\tfrac{1}{2}\sqrt{1+\tau+\tau^2} - \tfrac{1}{2}\left(1 + \tfrac{1}{2}\tau\right)\right]^{1/2}. \tag{17b}$$

Exercise 3. *Prove Equation* (17) *(not done in class).*

In view of the symmetric nature of the problem (sandwiched between $z = -1/2$ and $z = 1/2$), we can break up the solution into odd and even cases with respect to the centreline at $z = 0$. For the **even case** we have a solution

$$W = A_0 \cos(q_0 z) + A \cosh(qz) + A^* \cosh(q^* z),$$

where we have constructed a manifestly even solution via a linear superposition of component solutions. There are only three linearly independent complex coefficients in the superposition, and these can be chosen as A_0, A, and A^*, since A and A^* are linearly independent. Imposing the boundary conditions at $z = \pm 1/2$ yields

$$A_0 \cos(q_0/2) + A\cosh(q/2) + A^*\cosh(q^*/2) = 0, \tag{18a}$$

$$-q_0 A_0 \sin(q_0/2) + qA\sinh(q/2) + q^* A^* \sinh(q^*/2) = 0, \tag{18b}$$

$$A_0 \cos(q_0/2) + \tfrac{1}{2}\left(\mathrm{i}\sqrt{3} - 1\right) A\cosh(q/2) - \tfrac{1}{2}\left(\mathrm{i}\sqrt{3}+1\right) A^*\cosh(q^*/2) = 0. \tag{18c}$$

This immediately leads to a determinant problem

$$\begin{vmatrix} \cos(q_0/2) & \cosh(q/2) & \cosh(q^*/2) \\ -q_0\sin(q_0/2) & q\sinh(q/2) & q^*\sinh(q^*/2) \\ \cos(q_0/2) & \tfrac{1}{2}\left(\mathrm{i}\sqrt{3}-1\right)\cosh(q/2) & -\tfrac{1}{2}\left(\mathrm{i}\sqrt{3}+1\right)\cosh(q^*/2) \end{vmatrix} = 0. \tag{19}$$

We divide each row of the determinant problem by the first row to obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ -q_0\tan(q_0/2) & q\tanh(q/2) & q^*\tanh(q^*/2) \\ 1 & \tfrac{1}{2}\left(\mathrm{i}\sqrt{3}-1\right) & -\tfrac{1}{2}\left(\mathrm{i}\sqrt{3}+1\right) \end{vmatrix} = 0. \tag{20}$$

Next, we subtract the first row from the third row and divide the result by $-\sqrt{3}/2$ to obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ -q_0\tan(q_0/2) & q\tanh(q/2) & q^*\tanh(q^*/2) \\ 0 & \sqrt{3}-\mathrm{i} & \sqrt{3}+\mathrm{i} \end{vmatrix} = 0. \tag{21}$$

Expanding this determinant yields

$$\Im\left[\left(\sqrt{3}+\mathrm{i}\right)q\tanh(q/2)\right] + q_0\tan(q_0/2) = 0. \tag{22}$$

Exercise 4. *Fill in the blanks in the derivations of Equations* (18)*,* (21) *and* (22) *(not done in class).*

Since q and q_0 are both functions of k and Ra, Equation (22) can be regarded as a condition of the form

$$\Phi(\mathrm{Ra}, k) = 0,$$

where Φ is a function of two variables. This is the implicit equation of a curve in $\mathrm{Ra} - k$ space – the neutral curve. Note that the neutral curve is independent of the Prandtl number. Thus, to determine the onset of instability, the Prandtl number is irrelevant - only the wavenumber and the Rayleigh number matter. The aim of the remainder of this section is to compute the neutral curve numerically.

Remark 4. *Concerning the missing information in Equation* (18c) *alluded to in class.*

There are some missing steps in Equation (18c). Recall, this is the boundary condition $(\partial_z^2 - k^2)^2 W = 0$, where $W = A_0\cos(q_0 z) + A\cosh(qz) + A^*\cosh(q^* z)$. Look at the cosine component first, and consider

$$\begin{aligned}
(\partial_z^2 - k^2)^2\cos(q_0 z) &= \left(\partial_z^4 - 2k^2\partial_z^2 + k^4\right)\cos(q_0 z), \\
&= \left(q_0^2 + 2k^2 q^2 + k^4\right)\cos(q_0 z), \qquad \text{(note the sign)}, \\
&= (q_0^2 + k^2)^2\cos(q_0 z), \\
&= [(k^2\tau - k^2) + k^2]^2\cos(q_0 z), \qquad \text{as } q_0 = k^2(\tau - 1), \\
&= k^4\tau^2\cos(q_0 z).
\end{aligned}$$

Similarly, consider the cosh component $A\cosh(qz)$:

$$\begin{aligned}
(\partial_z^2 - k^2)^2\cosh(qz) &= (q^2 - k^2)^2\cosh(qz), \\
&= \left[\left(k^2 + \tfrac{1}{2}\tau k^2(1 + \mathrm{i}\sqrt{3})\right) - k^2\right]\cosh(qz), \\
&= \left(k^2\tau\omega\right)^2\cosh(qz),
\end{aligned}$$

where $\omega = (1 + \mathrm{i}\sqrt{3})/2$ and

$$\omega^2 = -\frac{1}{\omega} = \tfrac{1}{2}\left(\mathrm{i}\sqrt{3} - 1\right).$$

From these calculations, it is obvious that

$$\left(\partial_z^2 - k^2\right) W = (k^2\tau)^2 A_0 \cos(q_0 z) + (k^2\tau)^2 \left[\tfrac{1}{2}\left(\mathrm{i}\sqrt{3} - 1\right) A \cosh(qz) + +c.c.\right],$$

from which Equation (18c) follows.

We write a Matlab code to solve for the neutral curve. First, for a given k-value, the critical Rayleigh number can be estimated as follows:

```
function Ra=my_rayleigh_benard0(k,Ra_guess)

Ra=fzero(@myfun,Ra_guess);

function y=myfun(x)

    tau=(x/k^4)^(1/3);
    q0=k*sqrt(tau-1);
    temp=sqrt(1+tau+tau^2)+(1+(1/2)*tau);
    q1=k*sqrt(temp/2);
    temp=sqrt(1+tau+tau^2)-(1+(1/2)*tau);
    q2=k*sqrt(temp/2);

    q=q1+sqrt(-1)*q2;

    y=imag(  (sqrt(3)+sqrt(-1))*q*tanh(q/2)  )+q0*tan(q0/2);
end

end
```

my_rayleigh_benard0.m

Next, for a range of k-values, the corresponding set of critical Rayleigh numbers can be found as follows:

```
function [k_vec,Ra_vec]=my_rayleigh_benard1()

k_vec=0.1:0.05:10;
Ra_vec=0*k_vec;

Ra=my_rayleigh_benard0(k_vec(1),100000);
Ra_vec(1)=Ra;

for i=2:length(k_vec)
    Ra_guess=Ra_vec(i-1);
    Ra=my_rayleigh_benard0(k_vec(i),Ra_guess);
    Ra_vec(i)=Ra;
end

end
```

my_rayleigh_benard1.m

This code can be used to plot the neutral curve (Figure 6) for the **even case**.

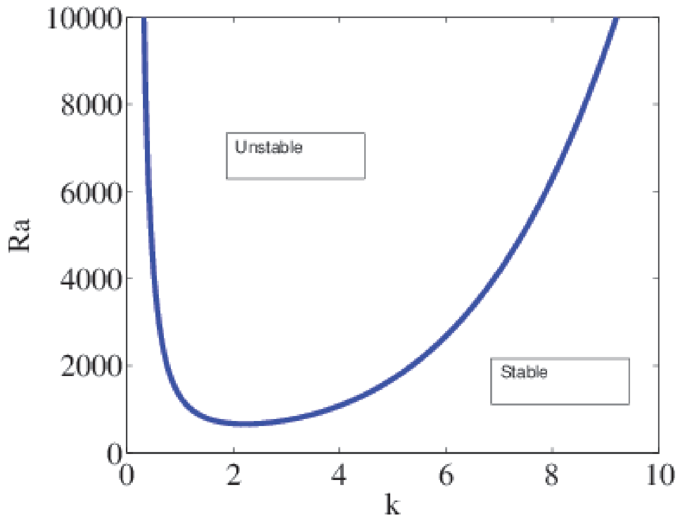

Figure 1: Neutral curve for the Rayleigh–Benard problem (even eigensolution)

In a similar way, the neutral curve for the odd mode is found from the root of the following equation:

$$q_0 \cot(q_0/2) = \Im\left[\left(\sqrt{3}+\mathrm{i}\right) q \frac{\sinh(q_1) - \mathrm{i}\sin(q_2)}{\cosh(q_1) - \cos(q_2)}\right]. \tag{23}$$

Exercise 5. *Starting with Equation* (23)*, write a Matlab function to construct the neutral curve of the odd eigensolution. Then, plot the odd and even neutral curves on a single graph. Show that the critical Rayleigh number for the onset of an even unstable eigenmode is less than the corresponding critical Rayleigh number for the odd unstable eigenmode. Argue then that the even eigenmodes are more unstable than the odd ones (not done in class).*

7 Convection patterns

We pass from two-dimensional to three-dimensional disturbances in the coodinates (x, y, z), where z is the wall-normal direction. It can be seen quite readily that the three-dimensional linearized equations of motion read

$$\frac{\partial}{\partial t}\nabla^2 w = +g\alpha\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) + \nu\nabla^4 w, \tag{24a}$$

$$\frac{\partial\theta}{\partial t} = w\beta + \kappa\nabla^2\theta \tag{24b}$$

where the differential operators are now in the appropriate three-dimensional form. Under a normal-mode decomposition

$$w = \mathrm{e}^{\mathrm{i}(k_x x + k_y y) + pt} W(z), \qquad \theta = \mathrm{e}^{\mathrm{i}(k_x x + k_y y) + pt} \Theta(z),$$

the eigenvalue equation derived previously **still persists**, only now the quantity k^2 in the relevant differential equation means $k^2 = k_x^2 + k_y^2$. Thus, there was no loss of generality in our previous focus on the two-dimensional case. Interestingly, the theory at this stage is by no means complete, since at the onset of **criticality** (i.e. for parameters along the neutral curve) there are many ways in which the critical wavenumber k_c^2 can be resolved into its x- and y-components. Thus, the theory so far does not tell us which pair (k_x, k_y) (with $k_x^2 + k_y^2 = k_\mathrm{c}^2$) is selected. Indeed, any pair consistent with this condition is possible and hence, a linear superposition of all such consistent pairs is the general acceptable solution.

However, we can observe that a particular wavenumber choice corresponds to a periodic cell, replicated throughout the xy-plane. Because the problem is translationally invariant in the xy-plane, these cells should fill in the xy plane with no gaps. There is a loose analogy here with solid-state physics: translational symmetry in a discrete crystal structure implies a lattice structure, which in turn implies that the only possibility for the unit cell (in two dimensions) is a square, an equilateral triangle, or a hexagon. Thus, **only those wavenumber combinations that produce a square, equilateral triangle or a hexagon as the periodic cell are allowed by the translational symmetry of the problem.** A real hexagonal convection cell is shwon in Figure 2.

The complete velocity field

Remark 5. *This part was not done in class.*

The complete velocity field (u, v, w) can be backed out from these considerations, albeit in a remarkably roundabout fashion. First, we note that

$$w = F(x, y) W(z),$$

where $F(x, y)$ is that combination of complex exponentials that gives the relevant periodic unit cell, such that

$$\left(\partial_x^2 + \partial_y^2\right) F = -k^2 F.$$

Because of its ubiquity in the following, we call

$$\nabla_\perp = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right),$$

hence $\nabla_\perp^2 F = -k^2 F$. Next, we introduce the wall-normal component of the vorticity,

$$\zeta = \hat{\boldsymbol{z}} \cdot \boldsymbol{\omega} = \partial_x v - \partial_y u.$$

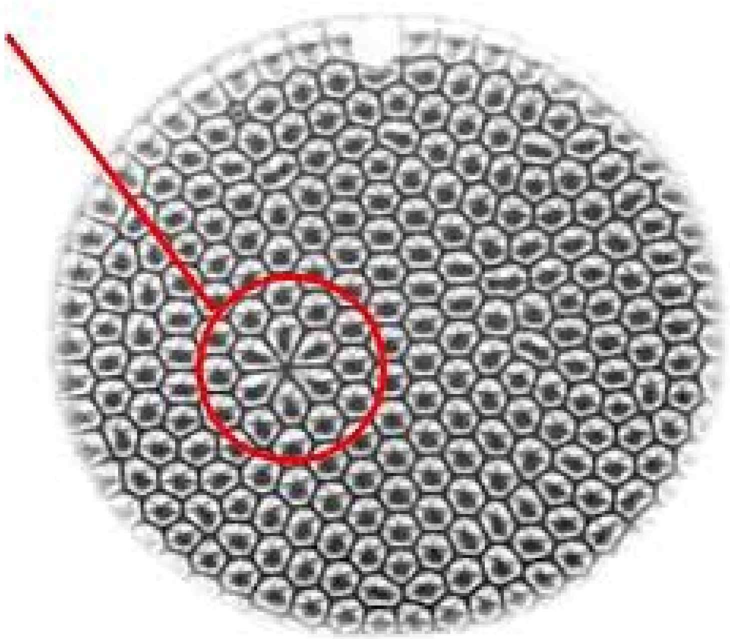

Figure 2: A (mostly) hexagonal array of convection cells in real-life Rayleigh–Bénard convection, from https://www.esrl.noaa.gov/psd/outreach/education/science/convection/RBCells.html, visited 02/02/2017.

We have

$$\frac{\partial \zeta}{\partial x} = \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y}, \tag{25a}$$

$$\frac{\partial \zeta}{\partial y} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2}. \tag{25b}$$

In view of the incompressibility condition $\partial_x u + \partial_y v + \partial_z w = 0$, we also have

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 w}{\partial y \partial z}, \tag{26a}$$

$$\frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 w}{\partial x \partial z}. \tag{26b}$$

We combine Equations (25)–(26) now to obtain

$$\frac{\partial \zeta}{\partial x} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial z} = \nabla_\perp^2 v + \frac{\partial^2 w}{\partial y \partial z} = -k^2 v + \frac{\partial^2 w}{\partial y \partial z},$$

hence

$$v = \frac{1}{k^2}\left(\frac{\partial^2 w}{\partial y \partial z} - \frac{\partial \zeta}{\partial x}\right).$$

Also,

$$\frac{\partial \zeta}{\partial y} = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 w}{\partial x \partial z} - \frac{\partial^2 u}{\partial y^2} = -\nabla_\perp^2 u - \frac{\partial^2 w}{\partial y \partial z} = k^2 u - \frac{\partial^2 w}{\partial y \partial z},$$

hence

$$u = \frac{1}{k^2}\left(\frac{\partial^2 w}{\partial x \partial z} + \frac{\partial \zeta}{\partial y}\right).$$

However, quite generally, we have the vortex stretching equation, which reads

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla u + \nu\nabla^2\boldsymbol{\omega},$$

the linearization of which is

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nu\nabla^2\boldsymbol{\omega},$$

and projecting on to the z-direction gives

$$\frac{\partial \zeta}{\partial t} = \nu\nabla^2\zeta.$$

In normal-mode form, this gives

$$\sigma\zeta = (\partial_z^2 - k^2)\zeta.$$

However, we are at criticality, with $\sigma = 0$, hence

$$(\partial_z^2 - k^2)\zeta = 0, \qquad \zeta = 0 \text{ on } z = \pm 1/2,$$

the only solution of which is $\zeta = 0$. Hence, the wall-normal component of the vorticity vanishes in this very particular case, and we are left with

$$u = \frac{1}{k^2}\frac{\partial^2 w}{\partial y \partial z}, \qquad v = \frac{1}{k^2}\frac{\partial^2 w}{\partial x \partial z}.$$

Letting $\boldsymbol{u}_\perp = (u, v)$, we have

$$\boldsymbol{u}_\perp = \frac{1}{k^2}\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}\right) = \frac{1}{k^2}W'\nabla_\perp F.$$

But $w = F(x, y)W(z)$, hence $F = w/W$, hence

$$\boldsymbol{u}_\perp = \frac{1}{k^2}\frac{W'}{W}\nabla_\perp w.$$

Thus, if the gradient $\nabla_\perp w$ vanishes, then so does $\boldsymbol{u}_\perp$. We now use these results to investigate the convection cells. We examine only two-dimensional rolls in depth here: the interested reader can study Chandrasekhar's book for an in-depth treatment of the three-dimensional structures: rectangular, triangular and hexagonal cells. Typical two-dimensional and three-dimensional convection cells are compared side-by side in Figure 3.

Convection rolls

Remark 6. *This part was done only very briefly in class.*

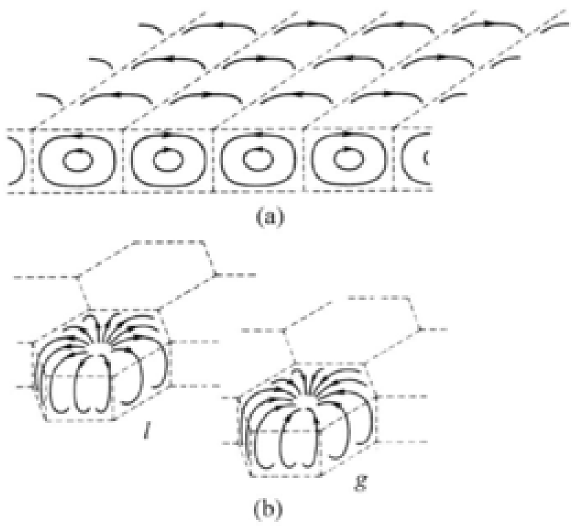

Figure 3: Typical two-dimensional and three-dimensional convection cells are compared side-by side. Schematic from Scholarpedia article on Rayleigh–Benard convection (accessed 07/01/2015).

The simplest convection pattern is the roll, wherein $k_y = 0$, and the problem reverts to a two-dimensional one. Let the critical wavenumber be k. Then , the size of the convection cell is $L = 2\pi/k$. The velocity profile is

$$w = W(z)\cos(kx), \qquad k = 2\pi/L$$

where $W(z)$ is the eigenfunction corresponding to the eigenvalue $\sigma = 0$. The corresponding components of the velocity parallel to the wall are

$$u = -\frac{1}{k}W'\sin(kx), \qquad v = 0.$$

It is clear that an appropriate streamfunction for the flow is

$$\psi = -\frac{1}{k}W\sin(kx),$$

with $u = \partial_z\psi$ and $w = -\partial_x\psi$. The streamlines can be plotted as isosurfaces of the streamfunction. For the W-component of the streamfunction, I use the approximation $W \approx z^2 - 2z^3 + z^4$, which satisfies the symmetry condition (even function) and boundary conditions but is otherwise an approximation of the true eigenfunction. The result of the plot is shown in Figure 4. The main feature here is two counter-rotating vortices in the cell that act to redistribute the temperature. This is the essential signature of Rayleigh–Bénard convection.

8 Beyond linear theory – the Nusselt Number

Beyond linear theory, the exponential growth of the convection rolls will either saturate (leading to steady, laminar flow) or themselves become unstable – in which case a pattern of turbulent

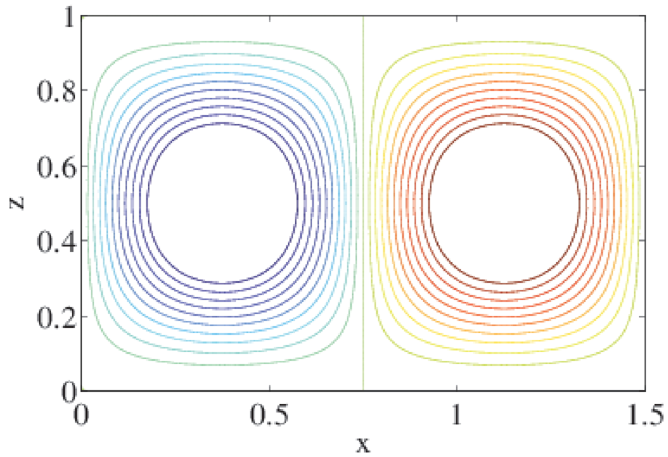

Figure 4: Two-dimensional convection roll in Rayleigh–Bénard convection (at criticality)

convection ensues. The eventual outcome of these processes depends on the Rayleigh number – the higher the Rayleigh number the less laminar the flow eventually is. In both cases, the vertical velocity represents a highly efficient means of transporting heat from bottom to top – over and above the heat transfer that can be achieved by diffusion alone. The enhancement is characterized by the Nusselt number:

$$\mathrm{Nu} = \frac{\langle \int_0^d \left(wT - \kappa \frac{\partial T}{\partial z} \right) \mathrm{d}z \rangle}{\kappa (T_{\mathrm{hot}} - T_{\mathrm{cold}})}, \tag{27}$$

where the angle brackets denote a space-time average (i.e. averaging over both time and the x- and y-directions), $w(x, y, z, t)$ is the instantaneous fluid velocity in the vertical (z-direction), and $T(x, y, z, t)$ is the corresponding instantaneous temperature field. Because these fields arise as solutions to the full (nonlinear) Navier–Stokes equations, there is no simple way to derive a closed form for the Nusselt number. However, some accepted correlations exist, which have been derived rigorously from decades of experiments and also, direct numerical simulations:

$$\mathrm{Nu} = \begin{cases} 0.54\mathrm{Ra}^{1/4}, & 10^3 \le \mathrm{Ra} \le 10^7, \qquad \mathrm{Pr} \ge 0.7, \\ 0.15\mathrm{Ra}^{1/3}, & 10^7 \le \mathrm{Ra} \le 10^{11}, \qquad \text{all } \mathrm{Pr} \end{cases} \tag{28}$$

The second of these correlations is the ‘classical’ Rayleigh–Bénard scaling, which applies when the convection rolls are fully turbulent. The scaling regime beyond $\mathrm{Ra} = 10^{11}$ is the subject of current research [2].

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