XIII.

Statistical Hydrodynamics. (*)

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It is a familiar fact of hydrodynamics, than when the « Reynolds number » exceeds a certain critical value, which depends on the type of flow, no steady flow is stable. The unsteady flow which occurs under these conditions calls for statistical analysis; but early attempts in this direction encountered formidable difficulties. Within the last few years, however, the most important remaining questions concerning the stability of laminar flow were settled by C. C. LIN [1], and a promising start towards a quantitative theory of turbulence was achieved by KOLMOGOROFF [2]. For good measure, KOLMOGOROFF's main result was rediscovered at least twice [3], [4]. The theories involved deal with the mechanism of turbulent dissipation. We shall return to this subject; it seems logical to discuss first a different, new application of statistics to hydrodynamics.

Ergodic Motion of Parallel Vortices.

The formation of large, isolated vortices is an extremely common, yet spectacular phenomenon in unsteady flow. Its ubiquity suggests an explanation on statistical grounds.

To that end, we consider n parallel vortices of intensities (circulations) $k_1, \ldots, k_n$ in an incompressible, frictionless fluid. This essentially two-dimensional system is Hamiltonian and has but a finite number (n) of degrees of freedom, so that we can apply the standard methods of statistical mechanics. The equations of motion may be written in the form

$$
(1) \qquad \begin{cases} k_i\, dx_i/dt = \partial H/\partial y_i\,, \\ k_i\, dy_i/dt = -\,\partial H/\partial x_i\,, \end{cases}
$$

where t denotes the time and H is the energy integral; the infinite self-energy

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of the individual vortices has been subtracted. In an unbounded fluid, H has the form [5, 6]:

$$(2)\qquad \begin{cases} H = -\dfrac{1}{2\pi}\sum\limits_{i>j} k_i k_j \log r_{ij}\ ; \\ r_{ij}^2 = (x_j - x_i)^2 + (y_i - y_j)^2\ . \end{cases}$$

The equations of motion (1) still apply when the liquid is restrained by boundaries, in which case the Hamiltonian (2) is modified so as to allow for image forces, and may be constructed in terms of the GREEN's functions of LAPLACE's equation [6].

Now let us consider the liquid enclosed by a boundary, so that the vortices are confined to an area A. We note that our dynamical system has some unusual properties. In effect, the x and y coordinates of each vortex are canonical conjugates, so that the phase-space is identical with the configuration-space of the vortices:

$$(3)\qquad d\Omega = dx_1\, dy_1 \ldots dx_n\, dy_n\ .$$

Moreover, this phase-space is finite

$$(4)\qquad \int d\Omega = \left(\int dx\, dy\right)^n = A^n\ .$$

The energy can assume all values from $+\infty$ (when two vortices of the same sign coincide) to $-\infty$ (when two vortices of opposite sign coincide, or when any one vortex is located on the boundary). The phase-volume which corresponds to energies less than a given value,

$$H(x_1, y_1, \ldots, x_n, y_n) < E\ ,$$

is a differentiable function of the energy:

$$(5)\qquad \begin{cases} \Phi(E) = \displaystyle\int_{H<E} d\Omega = \int_{-\infty}^{E} \Phi'(E)\, dE\ ; \\ \Phi(-\infty) = 0\ ; \\ \Phi(+\infty) = A^n\ . \end{cases}$$

Certainly $\Phi'(E)$ is positive for all E. Moreover, it must assume its maximum value for some finite energy E_m:

$$(6)\qquad \Phi''(E_m) = 0\ .$$

The temperature $\Theta = \Phi'/\Phi''$ will be positive

$$(7a)\qquad \infty > 1/\Theta \equiv \Phi''/\Phi' > 0\ ,$$

whenever:

$$(7b)\qquad E < E_m\ ,$$

but negative "temperatures"

$$0 > 1/\Theta > -\infty, \tag{8a}$$

can and will occur if

$$E > E_m. \tag{8b}$$

In the former case, vortices of opposite sign will tend to approach each other. However, if $1/\Theta < 0$, then vortices of the same sign will tend to cluster, — preferably the strongest ones —, so as to use up excess energy at the least possible cost in terms of degrees of freedom. It stands to reason that the large compound vortices formed in this manner will remain as the only conspicuous features of the motion; because the weaker vortices, free to roam practically at random, will yield rather erratic and disorganised contributions to the flow.

When we compare our idealised model with reality, we have to admit one profound difference: the distributions of vorticity which occur in the actual flow of normal liquids [1] are continuous, and in two-dimensional convection the vorticity of every volume element of the liquid is conserved, so that convective processes can build vortices only in the sense of bringing together volume elements of great initial vorticity. Thus our considerations would not apply to COUETTE flow, where the vortex density is constant, nor to POISEUILLE flow between parallel plates, nor to any other case of parallel flow in which the vortex density changes monotonically across the profile, so that no redistribution of vorticity is compatible with the conservation laws for energy and momentum. Until recently, the predicted stability of laminar flow for infinite REYNOLDS numbers in such cases was counted among the major puzzles of hydrodynamic theory; because all types studied experimentally become unstable at sufficiently high REYNOLDS numbers. The problem was solved when LIN [1] showed that viscosity and convection together lead to instability even when the vorticity has no extremum in the interior of the liquid.

This digression will make it clear that the present theory for the formation of large vortices does not apply to all cases of unsteady flow. As a matter of fact, the phenomenon is common but not universal. It is typically associated with separating boundary layers, whereby the initial conditions are not so very different from those contemplated in the theory: the vorticity is mostly concentrated in small regions, and the initial energy is relatively high.

From this cursory examination, it would seem that our highly idealized model has some heuristic value, although it must obviously be taken with a grain of salt at least. As a statistical model in two dimensions it is ambiguous: what set of discrete vortices will best approximate a continuous distribution

(1) Vortices in a suprafluid are presumably quantized; the quantum of circulation is h/m, where m is the mass of a single molecule.

of vorticity? Kinetic considerations (or tedious solution of hydrodynamic equations) may decide this question and one other: how soon will the vortices discover that there are three dimensions rather than two? The latter question is important because in three dimensions a mechanism for complete dissipation of all kinetic energy, even without the aid of viscosity, is available.

Turbulence.

The enhanced dissipation of energy which takes place in turbulent motion cannot be explained by any mechanism of two-dimensional convection. According to a well-known theorem of hydrodynamics, the total rate of dissipation is proportional to the "total vorticity":

$$(9)\qquad -\frac{d}{dt}\int \frac{1}{2}\rho v^2 dV = \eta \int |\operatorname{curl} \vec{v}|^2 dV ,$$

where t stands for the time, $\vec{v}$ for the velocity, ρ for density and η for viscosity, and the integration is extended over the entire volume of the fluid. Two-dimensional convection, which merely redistributes vorticity, cannot account for the rapid dissipation which one observes.

However, as pointed out by G. I. TAYLOR [7], convection in three dimensions will tend to increase the total vorticity. Since the circulation of a vortex tube is conserved, the vorticity will increase whenever a vortex tube is stretched. Now it is very reasonable to expect that a vortex line — of any line which is deformed by the motion of the liquid — will tend to increase in length as a result of more or less haphazard motion. This process tends to make the texture of the motion ever finer, and greatly accelerates the viscous dissipation. Experience indicates that for large REYNOLDS numbers the over-all rate of dissipation is completely determined by the intensity $\bar{v}^2$ together with the "macro-scale" L of the motion, and that the viscosity plays no primary role except through the condition that the REYNOLDS number

$$(10)\qquad \mathfrak{R} = (\bar{v}^2)^{1/2}\rho L/\eta$$

must be sufficiently large. Under the circumstances, dimensional considerations uniquely determine the law of dissipation

$$(11)\qquad Q = -\frac{d}{dt}\bar{v}^2 = (\text{const.})\,(\bar{v}^2)^{3/2}/L ,$$

and this has been verified by many experiments [8]. The concept of "macro-scale" is somewhat vaguely defined. It may be described in terms of the correlation between the fluctuations of velocity at neighbouring points in the fluids. If we define a correlation-function $R(r)$ thus

$$(12)\qquad \overline{(\vec{v}(\vec{r}\,')\cdot\vec{v}(\vec{r}\,'+\vec{r}))} = \bar{v}^2 R(r) ,$$

then the "macroscale" L may be defined as the distance beyond which $R(r)$ is less than some judiciously assigned value. There seems to be a general feeling that at the present state of our knowledge it would be premature to seek a much more precise definition.

Such a familiar type of turbulence as exists in a liquid flowing through a cylindrical tube is neither homogeneous nor isotropic. The mean fluctuations of the velocity vary over the cross-section of the tube, the local macroscale is generally comparable to the distance from the wall, and fluctuations as well as correlations are more or less anisotropic [9]. It is possible, however, to produce nearly homogeneous and isotropic turbulence by means of a grid in a streaming gas, and the "macroscale" is then predetermined by the mesh of the grid. In the following we shall assume that we are dealing with this simplest type of turbulence, which has been the object of extensive experimental studies.

According to the equations of motion:

$$\text{(13)} \qquad \begin{cases} \partial \vec{v}/\partial t = -(\vec{v}\cdot\vec{\nabla})\vec{v} - (1/\rho)\vec{\nabla}P + (\eta/\rho)(\vec{\nabla}\cdot\vec{\nabla})\vec{v}\ ; \\ (\vec{\nabla}\cdot\vec{v}) = 0\ , \end{cases}$$

the change of the correlation-function $R(r)$ with the time depends on the correlations between the velocities and the velocity gradients at two different points, etc. For reasons which will be more or less apparent in the following, a direct step-wise analysis of point-to-point correlations does not seem promising, however. An appropriate scheme of computation, quite possibly in terms of weighted averages, remains to be developed. Before we can arrive at a completely self-contained theory we shall have to determine somehow, from the laws of dynamics, a statistical distribution in function-space, and for the time being we do not know enough about how to describe such distributions.

Some important fundamental results concerning the distributions of energy in turbulent flow were nevertheless obtained by KOLMOGOROFF [2]. A fairly extensive literature is available, and a complete review would be out of place on this occasion. Rather I shall try to present the main line of reasoning in the simplest possible terms, along with some observations which may serve to supplement previous work, either to strengthen the argument or to bring out the significance of the results.

So as to give a more precise meaning to the concept of "scale", we describe the velocity field $\vec{v}(\vec{r})$ by its FOURIER series:

$$\text{(14)} \qquad \vec{v}(\vec{r}) = \sum_{\vec{k}} \vec{a}(\vec{k}) \exp(2\pi i \vec{k}\cdot\vec{r})\ , \qquad \vec{a}(-\vec{k}) = (\vec{a}(\vec{k}))\ \text{conjugate}\ ,$$

$$\text{(14a)} \qquad (\vec{k}\cdot\vec{a}(\vec{k})) = 0\ .$$

If the total volume is V, then there are $4\pi V k^2\, dk$ admissible wave-numbers $\vec{k}$

in the range of absolute values k, $k + dk$. As regards any attempt to bring about equiparation of the energy, each one of the FOURIER coefficients in (14) represents two degrees of freedom (not three, in view of (13a), viz. its equivalent (14a)), and since the total number is infinite, we foresee some kind of a "violent catastrophe". The kinetics of this process is described by the FOURIER transform of (13):

$$(15) \qquad d\vec{a}(\vec{k})/dt = 2\pi i \sum_{\vec{k}'} (\vec{a}(\vec{k}-\vec{k}')\cdot\vec{k}')\{-\vec{a}(\vec{k}') + (|\vec{k}|^2)^{-1}(\vec{a}(\vec{k}')\cdot\vec{k})\vec{k}\} - (\eta/\rho)\,|2\pi\vec{k}|^2\,\vec{a}(\vec{k})\,.$$

In this notation the rate of change of the distribution of energy is described as follows:

$$(16) \qquad d\,|\vec{a}(\vec{k})|^2/dt = -8\pi^2(\eta/\rho)\,|\vec{k}|^2\,|\vec{a}(\vec{k})|^2 + \sum_{\vec{k}'} Q(\vec{k}, \vec{k}')\,,$$

whereby the term:

$$(16a) \qquad Q(\vec{k}, \vec{k}') = \pi i\{(\vec{a}(\vec{k}+\vec{k}')\cdot\vec{k}')(\vec{a}(-\vec{k})\cdot\vec{a}(-\vec{k}')) + (\vec{a}(-\vec{k}+\vec{k}')\cdot\vec{k}')(\vec{a}(\vec{k})\cdot\vec{a}(-\vec{k}'))\} + (\text{conjugate})\,,$$

measures the net rate of transfer of energy from the wave-numbers $\pm\vec{k}$ to the wave-numbers $\pm\vec{k}'$. Since,

$$(17) \qquad Q(\vec{k}, \vec{k}') + Q(\vec{k}', \vec{k}) = 0\,,$$

the energy is indeed accounted for on both ends of the transaction.

We note that $\sum|\vec{a}(\vec{k})|^2$ is a measure for the total energy. Moreover, $\sum|\vec{k}|^2\,|\vec{a}(\vec{k})|^2$ is a measure for the total vorticity and for square of the over-all rate of deformation (practically the same quantity), and the rate of viscous dissipation is proportional to this. We realise that the viscous dissipation will consume the energy ever more readily as it is redistributed over an increasing range of wave-numbers.

In order to understand the law of dissipation described by (11), which does not involve the viscosity at all, we have to visualize the redistribution of energy as an accelerated cascade process. If we write the right member of (11) in the form of a product:

$$\bar{v}^2(\bar{v}^2/L^2)^{1/2}\,;$$

then the first factor represents the energy density, and most of this belongs to wave-numbers of the order $1/L$. The second factor is a rate of shear — not the over-all rate of deformation of the fluid, but only that part of it which belongs to motion on the largest scale. Indeed, it is not difficult to see that the modification of a smooth current by a fine-grained disturbance will depend on the total displacements involved, rather than on the rate of shear.

Now we note that according to (16a) the exchange of energy between wave-numbers $\pm\vec{k}$ and $\pm\vec{k}'$ depends only on the amplitudes $\vec{a}$ which belong to these wave-numbers and to their differences $(\pm\vec{k}\pm\vec{k}')$. If the latter, as well as $\vec{k}$ itself, are of the order $1/L$, then $\vec{k}'$ is at most of the order $2/L$. Similar reasoning may be applied to subsequent steps in the redistribution process, and we are led to expect a cascade such that the wave-numbers increase typically in a geometric series, by a factor of the order 2 per step. The energy is reprocessed through a given range of wave-numbers mainly with the aid of velocity gradients which belong to wave-numbers of the same order of magnitude.

The empirical law (11) suggests that the first few steps in the cascade limit the over-all speed of the process, which means that the subsequent steps must be accelerated. If so, then we may expect that the density of energy in the later stages will be determined by the rate at which energy is being handed down and ultimately dissipated. If we describe the distribution of energy in isotropic turbulence by the function:

$$\Omega(k) = 4\pi k^2 V \overline{|\vec{a}(\vec{k})|^2}, \tag{18}$$

then

$$\bar{v}^2 = \int_0^\infty \Omega(k)\, dk, \tag{18a}$$

and if $\Omega(k)$ is to be determined by the rate of dissipation, then the form of the distribution follows uniquely from dimensional considerations:

$$\Omega(k) = \beta Q^{2/3} k^{-5/3}, \tag{19}$$

where β is a dimensionless universal constant. Similarly we find a scale dependent characteristic time

$$t(k) = Q^{-1/3} k^{-2/3}. \tag{20}$$

The integral

$$\int_{1/L}^\infty t(k)\, dk/k < \infty, \tag{21}$$

converges, as required by our fundamental hypothesis of an accelerated cascade process. The scale-dependent coefficient of diffusion:

$$D(k) = (\text{const.})\, Q^{1/3} k^{-4/3}, \tag{22}$$

and the corresponding modified law of brownian motion

$$|\vec{r}_2 - \vec{r}_1|^2/t^3 = \text{const.}, \tag{23}$$

have been inferred long since from studies of natural turbulence, with the aid of commonplace indicators such as a rising column of smoke or a pair of small floating objects on the surface of the sea.

By the theory of FOURIER transforms, the distribution law (19) for the energy implies a corresponding form of the correlation-function

$$\bar{v}^2 R(r) = \bar{v}^2 - (\beta/3\Gamma(2/3))(Qr)^{2/3}, \tag{24}$$

valid for distance r appreciably smaller than the macroscale L yet greater than a certain "microscale"

$$\lambda \sim (\eta/\rho)^{3/4} Q^{1/4}, \tag{25}$$

where the viscous dissipation becomes dominant. The formulas (19), (20), (22) and (23) are of course subject to analogous limitations on both ends of the scale.

It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such "ideal" turbulence cannot obey any LIPSCHITZ condition of the form

$$|\vec{v}(\vec{r}' + \vec{r}) - \vec{v}(\vec{r}')| < (\text{const.})\, r^n, \tag{26}$$

for any order n greater than $1/3$; otherwise the energy is conserved. Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description; for example, the formulation (15) in terms of FOURIER series will do. The detailed conservation of energy (17) does not imply conservation of the total energy if the number of steps in the cascade is infinite, as expected, and the double sum of $Q(\vec{k}, \vec{k}')$ converges only conditionally.

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DISCUSSIONE E OSSERVAZIONI

Prof. M. BORN, *Edinburgh*:

Asks if ONSAGER's theory allows to calculate REYNOLD's number.

Prof. L. ONSAGER, *New Haven, Conn.*:

No, the problem of the REYNOLD's number is more complicated. Consult recent work of C. C. LIN.