Global Formal Series Solution of a Singularly Perturbed First Order Partial Differential Equation: A Problem in Statistical Mechanics (Version 8 – Preliminary)

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Abstract

We establish the existence of a unique global formal power series (in $1/N$) solution of a evolution equation corresponding to the renormalization group transformation of a continuum hierarchical N–Vectorial Heisenberg model in the sense that its Gevrey estimates holds uniformly for all $t > 0$. We also establish 1-summability of the unique formal power series for the initial condition which is related to the Fourrier transform of the uniform measure on the N–dimensional sphere.

1 Introduction and Motivations

The Set Up

In his joint work with C. Thompson [KT], M. Kac has assigned a uniform measure $\sigma_N(x_j; r)$ for each spin x_j lying on the N–dimensional sphere of radius $r = \sqrt{N}$, j running over a finite subset Λ of the d–dimensional lattice $\mathbb{Z}^d$, and by taking N to infinite, has inferred its thermodynamic properties from a well known model exhibiting phase transition. Kac-Thompson asymptotic analysis will be approached here via an evolution equation with initial data given by the characteristic function of $\sigma_N(x; \sqrt{N})$, whose orbits will be investigated for large N.

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The law of a spin sample $\mathbf{x} = (x_j)_{j\in\Lambda}$ in equilibrium with a reservoir at inverse temperature β is dictated by the Gibbs measure in $\mathbb{R}^{nN}$ $(n = card(\Lambda))$

$$\mu_{N,\Lambda}(\mathbf{x}) = \frac{1}{Z_{N,\Lambda}} e^{-\beta E_\Lambda(\mathbf{x})} \prod_{j\in\Lambda} \sigma_N(x_j; \sqrt{N}) \tag{1.1}$$

with the two-body interacting energy $E_\Lambda(\mathbf{x}) = -(\mathbf{x}, J\mathbf{x})_\Lambda/2 = -\sum_{i,j\in\Lambda} J_{ij} x_i \cdot x_j/2$ depending on the ferromgnetic ($J_{ij} \geq 0$) interacting matrix $J = [J_{ij}]$; $Z_{N,\Lambda}$ makes $\mu_{N,\Lambda}$ a probability measure. Kac-Thompson[KT] have shown that, for suitable J, every macroscopic quantity, usually an expected value with respect to (1.1) of a function of the spin average $\dfrac{1}{n}\displaystyle\sum_{j\in\Lambda} x_j$, approaches a limit as n and N goes to infinity (in any order); these limits is given by the same quantity evaluated in the correponding Berlin–Kac spherical model[BeK], provided β is not at the critical point β_c of this model (see [CM], for a friendly review).

The Berlin–Kac model is the simplest model displaying phase transition whose thermodynamic quantities can be easily evaluated using Laplace asymptotic method. In the present work we shall deal with the so called free energy function

$$f_\Lambda(z) = \int e^{i(\mathbf{z},\mathbf{x})_\Lambda} d\mu_{N,\Lambda}(\mathbf{x}) \tag{1.2}$$

in the presence of a uniform "imaginary field" $\mathbf{h} = i\mathbf{z} = (iz_j)_{j\in\Lambda}$, $z_j = z/n^{\gamma/2} \in \mathbb{R}^N$ (the Fourier transform of the equilibrium measure) in the thermodynamic limit $f(z) = \lim_{\Lambda\uparrow\mathbb{Z}^d} f_\Lambda(z)$, for large N. One should note that the macroscopic quantity (1.2) can be written as

$$f_\Lambda(z) = \mathbb{E}\exp\left(iz\cdot\left(\frac{1}{n^{\gamma/2}}\sum_{j\in\Lambda} x_j\right)\right) ,$$

with $\mathbb{E}$ the expectation w.r.t. $\mu_{N.\Lambda}$, and it is related with the spin (abnormally) averaged ($\gamma = d+2$ if β is at the critical point). Since $\mu_{N,\Lambda}(\mathbf{x})$ is invariant with respect to simultaneous rotations: $x_j' = Rx_j$, with R the same $N\times N$ orthogonal matrix for all $j\in\Lambda$, $f(z)$ actually depends on $|z|^2$ and we shall denote the derivative of f with respect to the variable $\xi = -|z|^2$ by the same letter f.

Kac pioneering work has sparked investigations in many directions and one may ask whether the formal series $\hat{f}$ in $1/N$ for a macroscopic quantity (and correlation function as well) is an asymptotic expansion of the analytic function f or, furthermore, whether $\hat{f}$ is Borel summable. We refer to Kupiainen's papers [K1, K2] (and references therein) for the $1/N$ asymptotic expansion at $\beta\neq\beta_c$ and the paper by Fröhlich, Mardin and Rivasseau [FMR] for Borel summability at sufficiently small β.

The PDE Equation

The coupling matrix J adopted in the above quoted investigations is the usual finite difference Laplacean

$$(-\Delta_{\text{diff}}\mathbf{x})_i = \sum_{j:|i-j|=1} (x_i - x_j)$$

(with certain boundary condition on Λ), under which only spins located at nearest neighbor vertices are coupled. To give detailed information on the formal series $\hat{f}$ at the critical point, we shall consider a hierarchical coupling matrix J of the type introduced by Dyson[D] (see [CM]). We refer to [GK, Ko] for previous investigation and to Watanabe's work [W] for a more recent and closely related to the present one. See also [AHMo] for investigation on the hierarchical spherical model.

A hierarchical Laplacean $-\Delta_{\text{hier}}$ in $\mathbb{Z}^d$, also called self-similar Laplacean, may be defined as follows (see e.g. [Mo, W, CM]). For a given integer $L > 1$ we define the block operator $B : \mathbb{R}^{\Lambda_K} \longrightarrow \mathbb{R}^{\Lambda_{K-1}}$

$$(Bx)_{\mathbf{i}} = \frac{1}{L^{d/2}} \sum_{j \in \{0,1,\ldots,L-1\}^d} x_{Li+j}$$

and its adjoint $B^* : \mathbb{R}^{\Lambda_{K-1}} \longrightarrow \mathbb{R}^{\Lambda_K}$ by $(B^*\mathbf{x}, \mathbf{y}) = (\mathbf{x}, B\mathbf{y})$, we have

$$-\Delta_{\text{hier}} = (L-1) \sum_{k=1}^{\infty} L^{-2k} \left(I - (B^*)^k B^k \right) = \frac{L-1}{L^2-1} I - J \ .$$

The invariance of $E(\mathbf{x}) = (\mathbf{x}, J\mathbf{x})/2 = (L-1)\sum_{k=1}^{\infty} \left(B^k\mathbf{x}, B^k\mathbf{x}\right)/2$ under the block operation allows us to study macroscopic quantities by investigating dynamical properties of a so called renormalization group transformation, defined on the space of measures in $\mathbb{R}^N$:

$$\sigma_k(x) = \frac{1}{C_k} e^{(L-1)|x|^2/2} \underbrace{\sigma_{k-1} * \cdots * \sigma_{k-1}}_{L^d\text{- times}} (L^{d/2+1}x) \tag{1.3}$$

$k = 1, 2, \ldots$, with the initial measure $\sigma_0(x) = \sigma_N(x, \sqrt{\beta N})$ uniform on the N–dimensional sphere of radius $\sqrt{\beta N}$.[1] Here $\delta * \eta$ stands for the convolution of two measures δ and η and C_k normalizes the measure.

In the present paper we shall instead consider a continuous version of this transformation introduced by Felder [F] (see [MCG], for details). If

$$\phi_k(z) = \int_{\mathbb{R}^N} \exp\left(ix \cdot z\right) d\sigma_k(x)$$

denotes the characteristic function (Fourier transform) of the measure σ_k and

$$U(t, z) = -\log \phi_k(z)$$

[1]For convenience, we have changed variables and include the inverse temperature β in the initial measure.

is defined in the double limit as $k \to \infty$ and $L \downarrow 1$ with $t = k \ln L$ kept fixed, we end up with the folowing initial value problem in $\mathbb{R}_+ \times \mathbb{R}^N$

$$U_t = -\frac{1}{2}\left(\Delta U - |U_z|^2\right) - \frac{d+2}{2} z \cdot U_z + dU + \frac{1}{2}\Delta U(t,0) \tag{1.4}$$

with $U(0,z) = -\ln \phi_0(z) \equiv U_0(z)$,

$$\phi_0(z) = \frac{\Gamma(N/2)}{\left(\sqrt{\beta N}|z|/2\right)^{N/2-1}} J_{N/2-1}\left(\sqrt{\beta N}|z|\right). \tag{1.5}$$

and J_ν the Bessel function of order ν. The last term in the right hand side of (1.4) ensures that $U(t,0) = 0$ for all $t \geq 0$ – note that this property is satisfied by the initial condition since $\phi_0^{(N)}(0) = 1$. Note also that $U^{\mathrm{Gauss}}(z) = |z|^2$ is an stationary solution of (1.4), corresponding to the Gaussian fixed point of the transformation (1.3), and the stable manifold associated with $U^{\mathrm{Gauss}}(z)$ has codimension 1 for $d \geq 4$.

This is the starting point of the present investigation. Our aim may be phrased as follows: *find a trajectory $\{U(t,z);\ t \geq 0\}$, for the initial condition problem with the inverse temperature β at the critical value β_c, so that it lies on the stable manifold and converges to the Gaussian fixed point $U^{\mathrm{Gauss}}(z)$.*

Since the initial function U_0 is far from the stationary solution U^{Gauss}, to accomplish our program we shall, as suggested by Kac, take N large.

Summary of Previous Results

The initial problem (1.4) and (1.5) has been studied in the limit $N \to \infty$ by the present authors in collaboration with Guidi [MCG]. For simplicity, the equation has been treated for $d = 4$, but all the results are valid for $d > 4$ as well and the analysis can be extended for any dimension $d > 2$ (see Remark 2.4 of [MCG]). Solving the critical trajectory, the following central limit theorem has then established for the hierarchical spherical model at critical inverse temperature $\beta_c(d=4) = 4$[2]

$$\lim_{t\to\infty} \lim_{N\to\infty} \frac{1}{N} U(t, \sqrt{N} z) = |z|^2 \quad .$$

The spherical symmetry is preserved by the equation so, it is enough to consider only the radial components of the gradient U_z in (1.4). Writing

$$u(t,x) = \frac{1}{N} U(t, \sqrt{N} z), \tag{1.6}$$

with $x = -|z|^2$, the initial value problem (1.4) and (1.5) reads

$$u_t = \frac{2}{N} x u_{xx} + u_x - 2x\left(u_x\right)^2 - (d+2) x u_x + du - u_x(t,0) \tag{1.7}$$

[2]The critical temperature may be found by Kac's asymptotic method (see [CM]). It can also be determined directly from the analysis in ([MCG]).

with
$$u(0,x) = \frac{1}{N} U_0(\sqrt{N} z)/N \equiv u_0(x) \ . \tag{1.8}$$

We see that it possesses $1/N$ in front of a second derivative term, in such way that the equation with finite N is a *singular perturbation* of the equation with $N = \infty$. This lead us to state our purpose as follows: *write the formal series $\hat{u}$ in $\varepsilon = 2/N$ of the unknown solution u as a singular perturbation about $\varepsilon = 0$ and recover via Borel summability the actual solution u (if it exists, it is analytic for each $t \geq 0$* in some polydomain*)*.

In [MCG], the initial problem (1.7) and (1.8) at $N = \infty$ has not been solved directly but its exact critical trajectory $\{u(t,x), t \geq 0, \beta = \beta_c(d)\}$ obtained solving the equation of its Legendre transform
$$w(t,p) = \max_x (xp - u(t,x)) \ . \tag{1.9}$$
There are certain advantages in pursuing the Legendre transformed solution. First of all, find the stable manifold of (1.7) is considerable difficult task, even in the neighborhood of $u^{\text{Gauss}}(x) = -x$. Surprisingly, β_c can be found for the $N = \infty$ equation as well as the critical trajectory of the perturbed formal series for $v(t,p) = w_p(t,p)$ (the inverse of $u_x(t,x)$) in any order of $2/N$. Moreover, the interpretation of the $N = \infty$ trajectory from the point of view of the geometric theory of functions allows to describe precisely the motion of the singularities of $u_x(t,x)$.

Indeed, the derivative of the initial condition (with $\varepsilon = 2/N$)
$$u_x(0,x) = \sqrt{\frac{\beta}{4x}} \frac{i\, J_{N/2}(i\sqrt{\beta x}N)}{J_{N/2-1}(i\sqrt{\beta x}N)} \equiv y(\varepsilon, x) \tag{1.10}$$
has simple poles located at the negative real line (the zeros of the Bessel function $J_{N/2-1}$) with negative residues. In the limit $N \to \infty$, the poles becomes dense over the real segment $(-\infty, -1/(4\beta)]$, giving rise to a branch cut. For every N (including its limit), $u_x(0,\xi)$ belongs to the class of Pick functions, which means that they are analytic functions in the upper half-plane $\mathbb{H}$ with positive imaginary part (i.e., it maps $\mathbb{H}$ into itself). We mention that the Pick class is preserved by the equation obtained by taking the derivative of (1.7) with respect to x and $u_x(t,\zeta)$, $t \geq 0$, map the upper half–plane $\mathbb{H}$ conformally into a decreasing family of open convex sets in $\mathbb{H}$ (see Theorem 2.2 of [MCG])
$$u_x(t, \mathbb{H}) = \Omega_t \subset \Omega_0 = u_x(0, \mathbb{H})$$
(Ω_0 is the semi-circular domain $(\Re\xi + 2)^2 + (\Im\xi)^2 < 4$, $\Im\xi > 0$ and Ω_∞ is the interior of half-leaf Decartes' folium).

Figure 1

As $u_x\,(t,\zeta)$ is in the Pick class, it admits an integral representation (see e.g. [Do])

$$u_x\,(t,\xi) = -1 + \int_{-\infty}^{\infty} \left(\frac{1}{\lambda-\xi} - \frac{1}{\lambda-1/2} \right)\, d\mu(t,\lambda)$$

where $d\mu = \rho\, d\lambda$ is absolutely continuous Borel measure for $N=\infty$ and its support $\Sigma(t) = (-\infty, -d(t))$ moves entirely to infinity as $t\to\infty$, for β at the critical point β_c. Note that $u_x^{\mathrm{Gauss}} \equiv -1$.

Statement of Present Results

The present work is divided into two parts, each of them has its own circle of interests. In the first we address the initial condition $u(0,\varepsilon,\xi)$ of (1.7).[3] An ordinary differential equation (ODE) is deduced for $y(\varepsilon,\xi) = u_x(0,\varepsilon,\xi)$. Such equation depends on the inverse temperature β and under the following

Hypothesis 1.1 (β–hyp.) *There exist positive constants b and m such that $\beta=\beta(\varepsilon)$ is a real analytic function in a sectorial domain $S(0,\alpha;E)$, continuous in its closure $\bar S(0,\alpha;E)$, satisfing*

$$\left| \frac{1}{i!}\frac{d^i\beta(\varepsilon)}{d\varepsilon^i} \right| \le bK_i m^i \tag{1.11}$$

for all $i\ge 0$ and $\varepsilon\in\bar S(0,\alpha;E)$, where $K_0=K_1=1$ and $K_i = A(i-1)!^s/(i-1)$ for $i\ge 2$, $A>0$ a small number (see Lemma 2.7) and $s\ge 0$.[4] The derivatives $\beta^{(i)}/i!$ are, in addition, continuous when $\varepsilon\to 0$ in $S(0,\alpha;E)$ and we write

$$\beta^i = \lim_{S(0,\alpha;E)\ni\varepsilon\to 0} \beta^{(i)}(\varepsilon)/i!. \tag{1.12}$$

We show that:

(a) the ODE has an analytic solution $y(\varepsilon,\xi)$ in the domain $S(0,\gamma;E)\times D_\sigma(0)$,[5] where γ, E and σ are suitable constants. In additon, $y(\varepsilon,\xi)$ converges, as $\varepsilon\to 0$ in $S(0,\gamma;E)$, to the solution $y_0(\xi)$ of the ODE when $\varepsilon=0$, regular at $\xi=0$;

(b) the formal power series

$$\hat y(\varepsilon,\xi) = \sum_{i=0}^{\infty} y_i(\xi)\,\varepsilon^i \tag{1.13}$$

is of Gevrey order s', for every fixed $\xi\in\bar D_\sigma(0)$, where $s' := \max(1,s)$;

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[3] We shall frequently denote by $u(t,\varepsilon,\xi)$ the solution $u(t,x)$ of equation (1.7) with both $\Re\varepsilon = 2/N$ and $\Re\xi = x$ extended to the complex plane.

[4] We shall observe in Remark 4.2 that s is equal to 1 for the equation (1.14) with F given by (1.15).

[5] $D_\sigma(z_0)$ denotes the open disc of radius σ centered at z_0

(c) there exist positive constants C and μ such that $\left|\frac{1}{i!}\frac{\partial^i y(\varepsilon,\xi)}{\partial\varepsilon^i}\right| \leq Ci!^{s'}\mu^i$ holds for $i = 0, 1, 2, \ldots$ and $(\varepsilon, \xi) \in S(0, \gamma; E) \times \bar{D}_\sigma(0)$.

The statements **(a)**–**(c)** will then be used to conclude, at the end of Section 3, that the formal solution $\hat{y}(\varepsilon, \xi)$ is s'-summable in $\theta = 0$ direction (Theorem 3.7).

The **(a)**–**(c)** statements, together with the s'-summability, hold for more general ordinary differential equations of the form

$$\varepsilon\xi y' = F(\varepsilon, \xi, y) \ , \tag{1.14}$$

with $y = (y_1, \ldots, y_m)$ and $F = (F_1, \ldots, F_m)$ m–vector functions, F_i holomorphic in a polydisc, say $D_\rho(0) \times D_\rho(0) \times D_\rho(0)$, for some $\rho > 0$. As in ([BK]), the $m \times m$ matrix $A_0 = F_y(0, 0, 0)$ is assumed to be invertible, a condition that makes (1.14) to possess a regular singularity at $\xi = 0$, and every eigenvalue of A_0 has to be away from a sectorial domain, as well. Balser and Kostov have established **(a)**–**(c)** statements for linear systems of the form (1.14) with $F(\varepsilon, \xi, y) = A(\varepsilon, \xi)y + b(\varepsilon, \xi)$. We have extended their proof by introducing an extra ingredient (Lemma 2.7) to account for nonlinearity in the equation. The present work deals only with (1.14) for $m = 1$ and

$$F(\varepsilon, \xi, y) = -\frac{\beta(\varepsilon)}{2} - y + 2\xi y^2 \ , \tag{1.15}$$

the equation satisfied by the initial condition (1.10). The extension for $m > 1$ and any F will be presented in a separate paper.

Whether the same statements hold for the solution of (1.7) when $t > 0$ is a question addressed in the second part. Only items **(a)** and **(b)** will be dealt in the present work. The methods used for the initial condition do not apply for partial differential equations and other tools are needed to prove the corresponding **(a)** and **(b)** statements for the formal solution $\hat{u}_x(t, \varepsilon, \xi)$. In particular, s' needs to be larger than 3. Our results are presented in Section 4.

Outline of this Article

2 Preliminaries

In this section we present some definitions, a basic and important theorem and some tools which shall be of extreme importance in this work. We basically follow Paragraph 1 of [LMS].

2.1 Some Definitions

Formal Power Series Let $\mathcal{O}(r)$ denotes the ring of analytic functions on the closed disc $\bar{D}_r(z_0)$, the closure of $D_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. Let $\mathcal{O}(r)[[\varepsilon]]$ denotes the ring of formal power series in ε over the ring $\mathcal{O}(r)$, and we define $\mathcal{O}[[\varepsilon]]$ by

$$\mathcal{O}[[\varepsilon]] := \bigcup_{r>0} \mathcal{O}(r)[[\varepsilon]].$$

An element $\hat{f}(\varepsilon, z) \in \mathcal{O}[[\varepsilon]]$ is written as

$$\hat{f}(\varepsilon, z) = \sum_{i=0}^{\infty} f_i(z)\, \varepsilon^i,$$

where $f_i(z) \in \mathcal{O}(r)$ for some $r > 0$.

s-Gevrey Formal Power Series Let s be a nonnegative number. $\mathcal{O}(r)[[\varepsilon]]_s$, which is called of class Gevrey s or s-Gevrey for short, is the subring of $\mathcal{O}(r)[[\varepsilon]]$ whose coefficient satisfy, for some positive constants C and μ, the inequality

$$\max_{z \in \bar{D}_r(z_0)} |f_i(z)| \leq C i!^s \mu^i$$

for $i = 0, 1, \ldots$. We also define $\mathcal{O}[[\varepsilon]]_s$, the set of all power series in ε of Gevrey order s, by

$$\mathcal{O}[[\varepsilon]]_s := \bigcup_{r>0} \mathcal{O}(r)[[\varepsilon]]_s.$$

In particular, $\mathcal{O}[[\varepsilon]]_0$ is the set of all convergent power series in ε.

Gevrey Asymptotic Expansion Let $\theta \in \mathbb{R}$, $\alpha > 0$ and $0 < E \leq \infty$. We denote by $S(\theta, \alpha; E)$ a sectorial domain defined by

$$S(\theta, \alpha; E) := \{\varepsilon \in \mathbb{C} : |\arg \varepsilon - \theta| < \alpha/2, 0 < |\varepsilon| < E\}.$$

If the radius E is not so important to identify, we will suppress it and denote the sector by $S(\theta, \alpha)$. A sectorial domain S' is called a proper subsector of $S(\theta, \alpha; E)$ if its closure is contained in $S(\theta, \alpha; E) \cup \{0\}$.

Let $f(\varepsilon, z)$ be analytic on $\bigcap_{\alpha' < \alpha} S(\theta, \alpha') \times D_{r(\alpha')}(z_0)$, where $r(\alpha')$ may tend to 0 as $\alpha' \to \alpha$. We define:

Definition 2.1 *$\hat{f}(\varepsilon, z) \in \mathcal{O}[[\varepsilon]]_s$ is called a Gevrey asymptotic expansion of $f(\varepsilon, z)$ as $\varepsilon \to 0$ in $S(\theta, \alpha)$ if for any proper subsector $S' \subset S(\theta, \alpha; E)$ (with sufficiently small radius), there exist positive constants C, μ and $0 < r_1 < r$ such that*

(i) $\hat{f}(\varepsilon, z) \in \mathcal{O}(r_1)[[\varepsilon]]_s$,

(ii)

$$\max_{z\in\bar{D}_{r_1}(z_0)}\left|f(\varepsilon,z)-\sum_{i=0}^{I-1}f_i(z)\,\varepsilon^i\right|\le CI!^s\mu^I|\varepsilon|^I,\qquad \varepsilon\in S',\quad I=1,2,3,\ldots.$$

Definition 2.2 *An analytic function $f(\varepsilon,z)$ is said to be Gevrey order s asymptotic expandable in $S(\theta,\alpha)$ if it has a Gevrey asymptotic expansion $\hat{f}(\varepsilon,z)\in\mathcal{O}[[\varepsilon]]_s$.*

Let us denote by $\mathcal{A}^{(s)}(S(\theta,\alpha))$ the set of analytic functions which are Gevrey order s asymptotic expandable in $S(\theta,\alpha)$. We define a mapping

$$J:\mathcal{A}^{(s)}(S(\theta,\alpha))\longrightarrow\mathcal{O}[[\varepsilon]]_s,\tag{2.1}$$

where $Jf(\varepsilon,z)=\hat{f}(\varepsilon,z)$ is the Gevrey asymptotic expansion $\hat{f}(\varepsilon,z)$ of $f(\varepsilon,z)$.

Theorem 2.3

1. *[sector of narrow opening] The mapping J is surjective but is not injective for any $\theta\in\mathbb{R}$ and α with $\alpha\le\pi/s$.*
2. *[sector of wide opening] For any α with $\alpha>\pi/s$, the mapping J is not surjective but is injective for any $\theta\in\mathbb{R}$.*

s-Summability in a Direction θ

Definition 2.4 *Let $s>0$, $\theta\in\mathbb{R}$, and $\hat{f}(\varepsilon,z)\in\mathcal{O}[[\varepsilon]]$ be given. We say that $\hat{f}(\varepsilon,z)$ is s-summable in direction θ, if a sector $S(\theta,\alpha;E)$, with $\alpha>\pi/s$, and a function $f(\varepsilon,z)\in\mathcal{A}^{(s)}(S(\theta,\alpha;E))$ exist with $J(f(\varepsilon,z))=\hat{f}(\varepsilon,z)$. This $f(\varepsilon,z)$ is called the s-sum of $\hat{f}(\varepsilon,z)$ in direction θ.*

2.2 Important Tools and a Key Ingredient

Nagumo Norms Let $\mathcal{H}_r$ denotes the space of analytic functions on $D_r(z_0)$ and continuous in its closure $\bar{D}_r(z_0)$. For nonnegative integers k we define the *Nagumo norm of order k of a function $f\in\mathcal{H}_r$* by

$$\|f\|_k:=\sup_{z\in D_r(z_0)}(d_r(z))^k\,|f(z)|,\qquad\text{where}\ \ d_r(z)=r-|z-z_0|.\tag{2.2}$$

For $f,g\in\mathcal{H}_r$ and nonnegative integers k,l, the following properties hold:

1. $\|f+g\|_k\le\|f\|_k+\|g\|_k$;
2. $\|fg\|_{k+l}\le\|f\|_k\,\|g\|_l$;
3. $\|f'\|_{k+1}\le e(k+1)\|f\|_k$;

4. $\|f\|_k \leq r\|f\|_{k-1}$.

Furthermore, we have $|f(z)| \leq (d_r(z))^{-k}\, \|f\|_k$ and

5. $|f(z)| \leq (r-r')^{-k}\, \|f\|_k$ holds for every $z \in D_{r'}(z_0)$ with $r' < r$.

Excepting 3., the proof of these properties is elementary. In this work we use another property that makes $\mathcal{X}_{r,k} = (\mathcal{H}_r, \|\cdot\|_k)$ a scale of Banach spaces. Note that $f \in \mathcal{X}_{r,k}$ implies that $f \in \mathcal{X}_{r,l}$ for every $l > k$ and

$$\|f\|_l \leq \sup_{z\in D_r(z_0)} (r - |z-z_0|)^{l-k} \|f\|_k \leq r^{l-k}\|f\|_k \leq \|f\|_k$$

provided $r \leq 1$. Under this condition $\mathcal{X}_{r,k}$ is a subspace of $\mathcal{X}_{r,l}$, the injection $\mathcal{X}_{r,k} \longrightarrow \mathcal{X}_{r,l}$ is continuous and has norm ≤ 1.

Modified Nagumo Norms Consider now a function $f(\varepsilon, z)$ holomorphic in $S(\theta,\alpha;E)\times D_r(z_0)$, and its expansion

$$f(\varepsilon, z) = \sum_{n=0}^{\infty} a_n(\varepsilon)\,(z-z_0)^n.$$

For nonnegative integers k we define a modification of the Nagumo norms

$$\|f\|_k := \sup_{z\in D_r(z_0)} (d_r(z))^k \sum_{n=0}^{\infty} \sup_{\varepsilon\in S(\theta,\alpha;E)} |a_n(\varepsilon)|\,|z-z_0|^n.$$

With that definition, the properties listed for the Nagumo norms in the previous paragraph are still valid (see [BK]).

Nagumo Norms for Different Domains In Section 4 we shall need some properties relating Nagumo norms on different domains, $D_r(z_0)$ and $D_R(z_0)$, with $r < R$. To state them, we index the Nagumo norm (2.2) by its order k and domain radius r: $\|\cdot\|_{k,r}$.

Lemma 2.5 *If $0 < r < R < \infty$ and f is holomorphic in $D_R(z_0)$, then we have*

$$\|f\|_{k,r} < \|f\|_{k,R} \tag{2.3}$$

$$\|f'\|_{k,R} \leq \frac{1}{R-r}\,\|f\|_{k,r} \tag{2.4}$$

Proof We observe that

$$d_r(z) = r - |z-z_0| < R - |z-z_0| = d_R(z)$$

implies the inequality (2.3):

$$\sup_{z\in D_r(z_0)} (d_r(z))^k |f(z)| < \sup_{z\in D_r(z_0)} (d_R(z))^k |f(z)| \le \sup_{z\in D_R(z_0)} (d_R(z))^k |f(z)|$$

for $k \ge 0$. By Cauchy formula, for $z \in D_r(z_0)$,

$$\begin{aligned} |f'(z)| &= \left| \frac{1}{2\pi i} \int_{|z-\zeta|=s} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \right| \\ &\le \frac{1}{s} \max_{\zeta:|\zeta-z|=s} |f(\zeta)| \\ &\le \frac{1}{s} \|f\|_{k,r} \max_{\zeta:|\zeta-z|=s} d_r(\zeta)^{-k} \end{aligned} \tag{2.5}$$

holds with $0 < s < R - r$. To obtain Property 3. above, we use

$$d_r(\zeta) = r - |\zeta - z_0| \ge r - |z - z_0| - |\zeta - z| = r - |z - z_0| - s = d_r(z) - s$$

in (2.5) to get an upper bound for $d_r(\zeta)^{-k}$, and minimize it over s. We do here something different:

$$R - r > s = |z - \zeta| \ge |z - z_0| - |\zeta - z_0|$$

implies

$$d_R(z) = R - |z - z_0| > r - |\zeta - z_0| = d_r(\zeta)$$

which, when substituted into (2.5), yields (2.4) by choosing $s = R - r$.

$\square$

Remark 2.6 *By changing domain in the Nagumo norm, equation (2.4), contrarily to Property 3., keeps its order unchanged.*

Key Ingredient

Lemma 2.7 *Let $\lambda \ge 0$ be given and let A be a positive number such that*

$A \le \left(1 + \pi^2/3\right)^{-1}/2 = 0.1165536\ldots$. Consider the sequence $(C_l)_{l=0}^{\infty}$ with $C_0 = A$ and

$$C_l = \frac{A l!^{\lambda}}{l^2} \ , \qquad \forall\, l \ge 1 \ .$$

Then

$$\sum_{l=0}^{m} C_l\, C_{m-l} \le C_m \tag{2.6}$$

holds for every $m \ge 0$.

Proof Since $\binom{m}{l} \ge 1$,

$$\frac{1}{l} + \frac{1}{m-l} = \frac{m}{l(m-l)}$$

and $0 \le (a-b)^2 = 2(a^2+b^2) - (a+b)^2$ holds for any real numbers a and b, we have

$$\begin{aligned}
\frac{1}{C_m}\sum_{l=0}^{m} C_l\, C_{m-l} &\le A\left(2+\sum_{l=1}^{m-1}\frac{m^2}{l^2(m-l)^2}\right) \\
&\le 2A\left(1+\sum_{l=1}^{m-1}\left(\frac{1}{l^2}+\frac{1}{(m-l)^2}\right)\right) \\
&\le 2A\left(1+\frac{\pi^2}{3}\right) \le 1.
\end{aligned}$$

□

It thus follows from (2.6) that

$$\sum_{\substack{l_1,\ldots,l_k\ge 1:\\ l_1+\cdots+l_k=m}} C_{l_1}\cdots C_{l_k} \le C_m \tag{2.7}$$

holds for any $1 \le k \le m$, which will be used in Section .

Scott's Formula for High Order Chain Rule An alternative to Faà di Bruno formula for high order chain rule, due to Scott, is as follows (see e.g. [FLy]). For $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ and $n \ge 1$

$$(f\circ g)^{[n]} = \sum_{k=1}^{n} f^{[k]}\circ g \sum_{\substack{i_1,\ldots,i_k\ge 1\\ i_1+\cdots+i_k=n}} g^{[i_1]}\cdots g^{[i_k]} \tag{2.8}$$

where $h^{[n]}(w)$ stands for the n–th derivative of h with respect to w divided by $n!$: $h^{[n]}(w) = \dfrac{1}{n!}h^{(n)}(w)$.

3 The Initial Condition

We shall prove **(a)**–**(c)** statements of the Introduction and conclude by them (Theorem 3.7) that the formal solution $\hat{y}(\varepsilon,\xi)$ is s'-summable in $\theta = 0$ direction. The justification of Hypothesis 1.1 shall be given in Section 4, when we study the solution of (1.7) for $t > 0$.

3.1 ODE Satisfied by $u_0'(x)$

Using the abbreviations $\nu = 1/\varepsilon := N/2$, $k := \sqrt{\beta}N$, $r := i\sqrt{x}$ and $s := kr$, we write (1.8) as

$$u_0(x) = -\frac{1}{N}\ln\left[\frac{2^{\nu-1}\Gamma(\nu)}{s^{\nu-1}}J_{\nu-1}(s)\right]$$

where the Bessel function $J_{\nu-1}(s)$ of order $\nu - 1$ satisfies

$$(sJ'(s))' + \left(s - \frac{(\nu-1)^2}{s}\right) J(s) = 0 \ . \tag{3.1}$$

We set

$$\varphi(r) = \frac{2^{\nu-1}\Gamma(\nu)}{(kr)^{\nu-1}} J(kr)$$

and verify that this function satisfies, by (3.1), a second order differential equation

$$\varphi''(r) + \frac{2\nu - 1}{r}\varphi'(r) + k^2\varphi(r) = 0.$$

Finally, $u_0(x) = -\ln\varphi(i\sqrt{x})/N$ satisfies

$$4Nxu_0''(x) + 4N\nu u_0'(x) - 4N^2x\left(u_0'(x)\right)^2 + k^2 = 0 \ . \tag{3.2}$$

This, together with the above abbreviations, leads to the following statement:

Proposition 3.1 *With $\varepsilon := 2/N$, the analytic extension $y(\varepsilon, \xi)$ of the derivative $u_0'(x)$ of (1.8) satisfies a first order nonlinear equation*[6]

$$\varepsilon\xi y' + y - 2\xi y^2 + \frac{\beta}{2} = 0 \tag{3.3}$$

The equation with $\varepsilon = 0$ has two solutions

$$y_0^{\pm}(\xi) = \frac{-\beta_0}{1 \pm \sqrt{1 + 4\beta_0\xi}} \ , \tag{3.4}$$

where the function with plus sign is regular at $\xi = 0$ and belongs to the Pick class. We set $y_0 = y_0^+$.

See Subsection 3.2 of [MCG] for the statement about the Pick class.

3.2 Proof of Statement (a): Power Series in ξ

Lemma 3.2 *Let (3.3) be considered with $\beta = \beta(\varepsilon)$ obeying Hypothesis 1.1. There exist γ, E and σ such that (3.3) has a solution $y(\varepsilon, \xi)$ analytic in the domain $S(0, \gamma; E) \times D_\sigma(0)$. The solution $y(\varepsilon, \xi)$ converges, as $\varepsilon \to 0$ in the sector $S(0, \gamma; E)$, to the (plus sign) solution $y_0(\xi)$ of (3.3) when $\varepsilon = 0$.*

[6]Here $'$ means derivative with respect to x.

Proof Substituting the power series expansion

$$y(\varepsilon, \xi) = \sum_{n=0}^{\infty} a_n(\varepsilon)\, \xi^n \tag{3.5}$$

into (3.3), we are lead to a sequence of relations

$$\begin{aligned} a_0(\varepsilon) + \frac{\beta(\varepsilon)}{2} &= 0 & (3.6)\\ (\varepsilon n + 1) a_n(\varepsilon) - 2 \sum_{m=0}^{n-1} a_m(\varepsilon)\, a_{n-1-m}(\varepsilon) &= 0 & (3.7) \end{aligned}$$

for $n \geq 1$, such that for each fixed n, $a_n(\varepsilon)$ is uniquely determined in terms of earlier coefficients.

We observe that $(\varepsilon n + 1)^{-1}$ has poles at $\{-1/n, n \geq 1\}$ in the complex ε-plane. In addition, for all $n \geq 1$ and $\varepsilon \in S(0, \gamma; \infty)$, we have

$$|\varepsilon n + 1| > 1$$

if $0 < \gamma < \pi$, and

$$|\varepsilon n + 1| > \sqrt{1 - \cos^2 \gamma/2} \equiv \frac{1}{c} \tag{3.8}$$

if $\pi \leq \gamma < 2\pi$ and both inequalities hold for any $E > 0$. Note that the imposed condition $\gamma < 2\pi$ avoids the poles mentioned above.

Now, let α_l and b be the supremum in $S(0, \gamma; E)$ of $|a_l(\varepsilon)|$ and $|\beta(\varepsilon)|$, respectively, and suppose

$$\alpha_0 = \alpha C_0$$

and

$$\alpha_l \leq \alpha C_l \kappa^{-l}$$

holds for $l \geq 1$, with $(C_l)_{l \geq 0}$ the sequence in Lemma 2.7 with $\lambda = 0$. Hence, by (3.6) and Hypothesis 1.1, we have

$$\sup_{\varepsilon \in S(0,\gamma;E)} |a_0(\varepsilon)| = \alpha_0 = \frac{b}{2} = \alpha A, \tag{3.9}$$

and, by (3.7), (3.8), (2.6) and Hypothesis 1.1,

$$\begin{aligned} \alpha_n &\leq 2c \sum_{m=0}^{n-1} \alpha_m\, \alpha_{n-1-m} \\ &\leq 2c\alpha^2 \kappa^{-n+1} \sum_{m=0}^{n-1} C_m\, C_{n-1-m} \\ &\leq 2c\alpha^2 C_{n-1} \kappa^{-n+1} \leq \alpha C_n \kappa^{-n}, \end{aligned}$$

where the last inequality holds provided that (recalling (3.9))

$$\kappa \leq \frac{1}{8c\alpha} = \frac{A}{4cb}\,; \tag{3.10}$$

note $C_n/C_{n-1} = (1-1/n)^2 \geq 1/4$ for all $n \geq 2$. With α and κ being chosen in that way, i.e., satisfying (3.9) and (3.10), we conclude

$$\sup_{\varepsilon\in S(0,\gamma;E)} |a_l(\varepsilon)| = \alpha_l \leq \alpha\frac{A}{l^2}\kappa^{-l} \qquad \forall\, l \geq 1. \tag{3.11}$$

Therefore, the power series solution (3.5) of (3.3) converges uniformly in $S(0,\gamma;E)$ to an analytic function of $\xi \in \bar{D}_\sigma(0)$, with $\sigma < \kappa$. By (3.6) and (3.7), $\left(a_n(\varepsilon)\,\sigma^n\right)_{n\geq 0}$ is a sequence of analytic functions, uniformly bounded in $S(0,\gamma;E)$,

$$\sum_{n=0}^{\infty} \sup_{\varepsilon\in S(0,\gamma;E)} |a_n(\varepsilon)|\,\sigma^n \leq \frac{b\kappa}{2(\kappa-\sigma)}$$

and $y(\varepsilon,\xi)$ is, consequently, analytic in $S(0,\gamma;E)\times D_\sigma(0)$.

Moreover, from the uniform convergence of (3.5) we conclude that for fixed $\xi \in D_\sigma(0)$, the solution $y(\varepsilon,\xi)$ tends to

$$y(0,\xi) = \sum_{n=0}^{\infty} a_n(0)\,\xi^n = \frac{-\beta_0}{1+\sqrt{1+4\beta_0\xi}} = y_0(\xi)$$

as $\varepsilon \to 0$ in $S(0,\gamma;E)$, where $\beta_0 = \lim_{S(0,\gamma;E)\ni\varepsilon\to 0}$. Note that the solution $y_0^-(\xi)$ of (3.3) with $\varepsilon = 0$ is not regular at $\xi = 0$ and this concludes the proof of Lemma 3.2.

$\square$

3.3 Proof of Statement (b): Power Series in ε

Lemma 3.3 *Suppose the formal power series (1.13) satisfies equation (3.3), formally, with $\beta = \beta(\varepsilon)$ obeying Hypothesis 1.1. Then, the coefficients $(y_i(\xi))_{i\geq 0}$ of (1.13) are analytic functions of ξ in the open disc $D_{1/(4\beta_0)}(0)$ and there exist positive constants C and μ such that*

$$|y_i(\xi)| \leq C i!^{s'} \mu^i \tag{3.12}$$

holds for all $i \geq 0$ and $\xi \in \bar{D}_\sigma(0)$, with $s' := \max(1,s)$ and $\sigma < \kappa < 1/(4\beta_0)$. In other words, the formal power series $\hat{y}(\varepsilon,\xi) \in \mathcal{O}(\sigma)[[\varepsilon]]_{s'}$ is of Gevrey order s'.

Proof By hypothesis, the formal series

$$\hat{\beta}(\varepsilon) = \sum_{i=0}^{\infty} \beta_i\,\varepsilon^i \tag{3.13}$$

is of Gevrey order s. Substituting the formal power series (1.13) and (3.13) into (3.3), yield

$$y_0(\xi) - 2\xi(y_0(\xi))^2 + \frac{\beta_0}{2} = 0$$

and

$$\xi y'_{i-1}(\xi) + y_i(\xi) - 2\xi \sum_{j=0}^{i} y_j(\xi)\, y_{i-j}(\xi) + \frac{\beta_i}{2} = 0 \qquad .$$

for $i \geq 1$. Solving the first equation, choosing the function with plus sign (regular at $\xi = 0$), we have

$$y_0(\xi) = \frac{-\beta_0}{1 + \sqrt{1 + 4\beta_0\xi}} \, .$$

It follows from the second equation with $i = 1$

$$y_1(\xi) = \frac{1}{1 - 4\xi y_0(\xi)} \left\{ -\xi y'_0(\xi) - \frac{\beta_1}{2} \right\} \tag{3.14}$$

and with any $i \geq 2$

$$y_i(\xi) = \frac{1}{1 - 4\xi y_0(\xi)} \left\{ -\xi y'_{i-1}(\xi) + 2\xi \sum_{j=1}^{i-1} y_j(\xi)\, y_{i-j}(\xi) - \frac{\beta_i}{2} \right\} \tag{3.15}$$

and this relation determines uniquely $y_i(\xi)$ in terms of earlier coefficients.

Note that

$$1 - 4\xi y_0(\xi) = \sqrt{1 + 4\beta_0\xi}$$

is an analytic and nonvanishing function in $D_{1/(4\beta_0)}(0)$ and, therefore, $y_i(\xi)$ is analytic in the domain $D_{1/(4\beta_0)}(0)$. Since

$$\frac{1}{c_1} \equiv \sqrt{1 - 4\beta_0\kappa} \leq \left| \sqrt{1 + 4\beta_0\xi} \right| \leq \sqrt{1 + 4\beta_0\kappa} \equiv c_2 \tag{3.16}$$

holds for every $\xi \in \bar{D}_\kappa(0)$ and $\kappa < 1/(4\beta_0)$, we have

$$\sup_{\xi \in \bar{D}_\kappa(0)} |y_0(\xi)| = \sup_{\xi \in \bar{D}_\kappa(0)} \left| \frac{-\beta_0}{1 + \sqrt{1 + 4\beta_0\xi}} \right| = \frac{|\beta_0|}{\min\left(1 - 1/c_1, c_2 - 1\right)} \equiv c_3. \tag{3.17}$$

Now, to obtain an estimate on the growth rate of $|y_i(\xi)|$, we use the Nagumo norms (2.2) with $r = \kappa$.

Let us assume that

$$\|y_l\|_{l-1} \leq \delta C_l \nu^l \tag{3.18}$$

holds for $l = 1, 2, \ldots, i-1$with $C_l = Al!^\lambda / l^2$, for some positive constants λ, δ and ν to be determined. Then, it follows by (3.15), (3.16), (2.6), the properties of Nagumo norms and Hypothesis 1.1 that

$$
\begin{aligned}
\|y_i\|_{i-1} &\leq c_1 \left\{ \|\xi\|_0 \, \|y'_{i-1}\|_{i-1} + 2\|\xi\|_0 \sum_{j=1}^{i-1} \|y_j\|_{j-1} \, \|y_{i-j}\|_{i-j} + \frac{\|\beta_i\|_{i-1}}{2} \right\} \\
&\leq c_1 \left\{ \kappa e(i-1)\|y_{i-1}\|_{i-2} + 2\kappa^2 \sum_{j=1}^{i-1} \|y_j\|_{j-1} \, \|y_{i-j}\|_{i-1-j} + \frac{\kappa^{i-1}|\beta_i|}{2} \right\} \\
&\leq c_1 \left\{ \kappa e(i-1)\delta C_{i-1}\nu^{i-1} + 2\kappa^2\delta^2 C_i \nu^i + \frac{\kappa^{i-1} b K_i m^i}{2} \right\} \\
&\leq \delta C_i \nu^i,
\end{aligned}
\tag{3.19}
$$

where the last inequality holds provided

$$
c_1 \left\{ \frac{\kappa e}{\nu} \frac{i^2}{i^\lambda (i-1)} + 2\kappa^2 \delta + \frac{b}{2\kappa\delta} \frac{1}{(i-1)!^{\lambda-s}} \frac{i^2}{(i-1)i^\lambda} \left(\frac{\kappa m}{\nu} \right)^i \right\} \leq 1 \tag{3.20}
$$

is satisfied for all $i \geq 2$.

In order the first and last terms of (3.20) to be bounded, one sees that $\lambda \geq s' = \max(1, s)$; δ has to be sufficiently small in reason of the second term and $\nu > \kappa m$ large enough. So, we fix $\lambda = s'$ and choose first δ small and ν so large that (3.20) and $\|y_1\|_0 \leq \delta C_1 \nu$ hold. By (3.14) and (3.17),

$$
\begin{aligned}
\|y_1\|_0 &= \sup_{\xi \in D_\kappa(0)} \left| \frac{1}{1 - 4\xi y_0(\xi)} \left\{ -\xi y'_0(\xi) - \frac{\beta_1}{2} \right\} \right| \\
&\leq c_1 \left\{ 2\kappa c_1 c_3^2 + \frac{bm}{2} \right\} \leq \delta C_1 \nu
\end{aligned}
\tag{3.21}
$$

holds for δ small, by taking ν sufficiently large.

This together with (3.19) complete the induction: with δ and ν fixed so that (3.21) and (3.20) hold, we thus have

$$
\sup_{\xi \in D_\kappa(0)} (d_\kappa(\xi))^{l-1} |y_l(\xi)| \equiv \|y_l\|_{l-1} \leq \delta \frac{A l!^{s'}}{l^2} \nu^l \qquad \forall\, l \geq 1. \tag{3.22}
$$

By the properties 4 and 5 of Nagumo norms,

$$
|y_i(\xi)| \leq \frac{\kappa}{(\kappa - \sigma)^i} \|y_i\|_{i-1} \leq C i!^{s'} \mu^i
$$

holds for all $i \geq 1$ uniformly in $\bar{D}_\sigma(0)$ for some $\sigma < \kappa$, with $C = \kappa\delta A$ and $\mu = \nu/(\kappa - \sigma)$. In order to include the $i = 0$ case, it is enough to choose $C = \max(c_3, \kappa\delta A)$, in view of (3.17), which concludes the proof of Lemma 3.3.

□

3.4 Proof of Statement (c): Gevrey Asymptotics

Lemma 3.4 *If $\beta = \beta(\varepsilon)$ obeys Hypothesis 1.1 then, for each sector $S(0,\gamma;E)$ and closed disc $\bar{D}_\sigma(0)$, $\sigma < \kappa/(1+2bc\kappa)$, with γ and E as in Lemma 3.2 and b, c and κ are constants in its proof, there exist positive constants C and μ such that*

$$\left| \frac{1}{i!} \frac{\partial^i y(\varepsilon,\xi)}{\partial \varepsilon^i} \right| \le C i!^s \mu^i$$

holds for all $i \ge 0$.

Remark 3.5 *A subproduct of the proof of Lemma 3.4 is another representation of the unique solution $y(\varepsilon,\xi)$ of (3.3), analytic in the domain $S(0,\gamma;E) \times D_\sigma(0)$, continuous in its closure, such that $\lim_{S(0,\gamma;E)\ni\varepsilon\to 0} y(0,\xi) = y_0(\xi)$.*

Proof Set

$$\begin{aligned} \phi_i(\varepsilon,\xi) &= \frac{1}{i!}\frac{\partial^i y}{\partial \varepsilon^i}(\varepsilon,\xi) \\ \beta_i(\varepsilon) &= \frac{1}{i!}\beta^{(i)}(\varepsilon) \\ \phi_i'(\varepsilon,\xi) &= \frac{\partial \phi_i}{\partial \xi}(\varepsilon,\xi) \end{aligned}$$

for every $(\varepsilon,\xi) \in S(0,\gamma;E) \times D_\kappa(0)$. Differentiating (3.3) i–times with respect to ε, yields

$$\varepsilon\xi\phi_i' + \xi\phi_{i-1}' + \phi_i - 2\xi \sum_{j=0}^{i} \phi_j\, \phi_{i-j} + \frac{\beta_i}{2} = 0\ .$$

These relations, together with (3.3) itself ($\phi_0(\varepsilon,\xi) = y(\varepsilon,\xi)$), may be written as

$$\varepsilon\xi\phi_i' + A_i(\varepsilon,\xi)\phi_i = g_i(\varepsilon,\xi), \qquad i \ge 0 \tag{3.23}$$

where

$$A_i(\varepsilon,\xi) = \begin{cases} 1 - 2\xi\phi_0(\varepsilon,\xi) & \text{if} \quad i = 0 \\ 1 - 4\xi\phi_0(\varepsilon,\xi) & \text{if} \quad i \ge 1 \end{cases}\ , \tag{3.24}$$

$$g_0(\varepsilon,\xi) = -\frac{\beta_0(\varepsilon)}{2}\ , \tag{3.25}$$

$$g_1(\varepsilon,\xi) = -\xi\phi_0'(\varepsilon,\xi) - \frac{\beta_1(\varepsilon)}{2}\ , \tag{3.26}$$

and

$$g_i(\varepsilon,\xi) = -\xi\phi_{i-1}'(\varepsilon,\xi) + 2\xi \sum_{j=1}^{i-1} \phi_j(\varepsilon,\xi)\, \phi_{i-j}(\varepsilon,\xi) - \frac{\beta_i(\varepsilon)}{2} \tag{3.27}$$

for $i \geq 2$, depends only on derivatives with respect to ε of order lower than i. (3.23) is a linear singular perturbation equation with regular singularity which can be dealt with the following auxiliary result due to Balser-Kostov [BK]. For this, we drop temporarily all subindices i in (3.23).

Let

$$A(\varepsilon,\xi) - A_0(\varepsilon) = \sum_{n=1}^{\infty} A_n(\varepsilon)\,\xi^n = -B(\varepsilon,\xi) \tag{3.28}$$

and consider a sequence of solutions $(\psi_k(\varepsilon,\xi))_{k\geq 0}$ of the system

$$\begin{cases} \varepsilon\xi\psi_0'(\varepsilon,\xi) + \psi_0(\varepsilon,\xi) = g(\varepsilon,\xi) \\ \varepsilon\xi\psi_k'(\varepsilon,\xi) + \psi_k(\varepsilon,\xi) = B(\varepsilon,\xi)\psi_{k-1}(\varepsilon,\xi)\ , \qquad k = 1,2,\ldots \end{cases}, \tag{3.29}$$

Then, the sum over all equations in (3.29) yields, by (3.28) and linearity, an equation of the form (3.23) for the sum $\psi(\varepsilon,\xi)$ of solutions $(\psi_k(\varepsilon,\xi))_{k\geq 0}$. Note, by (3.5), (3.6) and (3.7), that $A_0(\varepsilon) = A(\varepsilon,0) = 1$ and

$$-B(\varepsilon,\xi) = 4\xi y(\varepsilon,\xi) = 4\sum_{n=1}^{\infty} a_{n-1}(\varepsilon)\,\xi^n, \tag{3.30}$$

so $A_n(\varepsilon) = -4a_{n-1}(\varepsilon)$ for $n \geq 1$ (when $i = 0$, we replace the 4's by 2). We assume that $g(\varepsilon,\xi)$ admits an expansion

$$g(\varepsilon,\xi) = \sum_{n=0}^{\infty} g_n(\varepsilon)\,\xi^n \tag{3.31}$$

uniformly convergent in $S(0,\gamma;E) \times \bar{D}_{\sigma_1}(0)$, $\sigma_1 < \kappa$. For g given by (3.25)-(3.27) this will, actually, be proven by induction when we resume the proof of Lemma 3.4. We write, in addition, $f(z) \ll F(z)$ if $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is majorized by $F(z) = \sum_{k=0}^{\infty} C_k z^k$, i. e., if $|c_k| \leq C_k$ holds for all k.

Lemma 3.6 *There exist unique functions $(\psi_k(\varepsilon,\xi))_{k\geq 0}$, holomorphic in $S(0,\gamma;E) \times \bar{D}_{\sigma_1}(0)$, satisfying (3.29). Each $\psi_k(\varepsilon,\xi)$ has a zero of order k at $\xi = 0$ and satisfies*

$$\sum_{k=0}^{\infty} \psi_k(\varepsilon,\xi) = \psi(\varepsilon,\xi) \ll \frac{cG(\xi)}{1 - c\Omega(\xi)}\,, \tag{3.32}$$

where

$$G(\xi) = \sum_{n=0}^{\infty} \sup_{\varepsilon\in S(0,\gamma;E)} |g_n(\varepsilon)|\,\xi^n\ , \tag{3.33}$$

$$\Omega(\xi) = \sum_{n=0}^{\infty} \sup_{\varepsilon\in S(0,\gamma;E)} |A_n(\varepsilon)|\,\xi^n \tag{3.34}$$

and $|\Omega(\xi)| < 1/c$ *provided* $\sigma_1 < \kappa/(1+2bc\kappa)$, *with* b, c *and* κ *as in (3.9), (3.8) and (3.10), respectively.* $\psi(\varepsilon,\xi)$ *is, in addition, the unique analytic solution in* $S(0,\gamma;E)\times D_{\sigma_1}(0)$ *of*

$$\varepsilon\xi\psi'(\varepsilon,\xi)+A(\varepsilon,\xi)\psi(\varepsilon,\xi)=g(\varepsilon,\xi). \tag{3.35}$$

Proof Plugging

$$\psi_k(\varepsilon,\xi)=\sum_{n=k}^{\infty}\varpi_{n,k}(\varepsilon)\,\xi^n$$

into (3.29), yields

$$\begin{aligned}(\varepsilon n+1)\varpi_{n,0}(\varepsilon) &= g_n(\varepsilon)\ , \qquad n\ge 0\\ (\varepsilon n+1)\varpi_{n,k}(\varepsilon) &= -\sum_{m=k-1}^{n-1}A_{n-m}(\varepsilon)\,\varpi_{m,k-1}(\varepsilon)\ , \qquad 1\le k\le n\ .\end{aligned}$$

From the first relation we have

$$\psi_0(\varepsilon,\xi)=\sum_{n=0}^{\infty}\frac{1}{\varepsilon n+1}h_n(\varepsilon)\,\xi^n,$$

and from the second one

$$\begin{aligned}\psi_k(\varepsilon,\xi) &= \sum_{n=k}^{\infty}\frac{-1}{\varepsilon n+1}\sum_{m=k-1}^{n-1}A_{n-m}(\varepsilon)\,\varpi_{m,k-1}(\varepsilon)\,\xi^n\\ &= \sum_{m=k-1}^{n-1}\varpi_{m,k-1}(\varepsilon)\,\xi^m\sum_{l=1}^{\infty}\frac{-1}{\varepsilon(m+l)+1}A_l(\varepsilon)\,\xi^l\ .\end{aligned}$$

Defining

$$\Psi_k(\xi)=\sum_{n=k}^{\infty}\sup_{\varepsilon\in S(0,\gamma;E)}|\varpi_{n,k}(\varepsilon)|\,\xi^n,$$

it follows, by (3.8), (3.33) and (3.34) that

$$\begin{aligned}\Psi_0(|\xi|) &\le cG(|\xi|)\\ \Psi_k(|\xi|) &\le c\Omega(|\xi|)\Psi_{k-1}(|\xi|)\end{aligned}$$

for $k\ge 1$. Since $\psi_k(\varepsilon,\xi)\ll\Psi_k(\xi)$ for $k\ge 1$ and $\psi_0(\varepsilon,\xi)\ll cG(\xi)$ for $k=0$ hold for all $(\varepsilon,\xi)\in S(0,\gamma;E)\times\bar{D}_{\sigma_1}(0)$, we conclude (3.32) provided the geometric series $S(\sigma_1)=\sum_{k\ge 1}c^k\Omega(\sigma_1)^k$ converges. By (3.28), (3.30), (3.11), and (3.9),

$$\Omega(\sigma_1)=4\sum_{n=1}^{\infty}\sup_{\varepsilon\in S(0,\gamma;E)}|a_{n-1}(\varepsilon)|\,\sigma_1^n\le\frac{2b\kappa\sigma_1}{\kappa-\sigma_1}<\frac{1}{c}$$

if $\sigma_1 < \kappa/(1+2bc\kappa)$ and thence, $\sum_{k=0}^{\infty} \psi_k(\varepsilon,\xi) = \psi(\varepsilon,\xi)$ is a uniformly convergent series of analytic functions in $S(0,\gamma;E) \times D_{\sigma_1}(0)$ which solves (3.35). Since no other solution of (3.35), regular at $\xi = 0$, exists, the proof of Lemma 3.6 is concluded.

□

We continue the proof of Lemma 3.4. It remains to show that the $g_i(\varepsilon,\xi)$, given by (3.25), (3.26) and (3.27), satisfy the hypothesis of Lemma 3.6. This follows by induction. Clearly, $g_0(\varepsilon,\xi)$ is holomorphic in $S(0,\gamma;E) \times D_{\sigma_1}(0)$. Suppose that $\phi_j(\varepsilon,\xi)$ is holomorphic in $S(0,\gamma;E) \times D_{\sigma_1}(0)$ for each $1 \le j < i$. Then, by (3.27), $g_i(\varepsilon,\xi)$, is holomorphic in the same domain. By Lemma 3.6, $\phi_i(\varepsilon,\xi)$ is holomorphic in $S(0,\gamma;E) \times D_{\sigma_1}(0)$ and, by (3.27), we conclude it also holds for $g_{i+1}(\varepsilon,\xi)$, justifying its representation as a convergent series (3.31), uniformly in $S(0,\gamma;E) \times D_{\sigma_1}(0)$. By induction, $\phi_i(\varepsilon,\xi)$ is holomorphic in $S(0,\gamma;E) \times D_{\sigma_1}(0)$ for each $i \ge 1$ and

$$\phi_i(\varepsilon,\xi) \ll \frac{cG_i(\xi)}{1-c\Omega(\xi)} \ll \frac{cG_i(\xi)}{1-c\Omega(\sigma_1)}\,. \tag{3.36}$$

where G_i depends on the $\phi_j(\varepsilon,\xi)$ with $j < i$. For $i = 0$, by (3.24), (3.25), (3.33) and Hypothesis 1.1,

$$|y_0(\varepsilon,\xi)| = |\phi_0(\varepsilon,\xi)| \le \frac{cG_0(|\xi|)}{1-c\Omega(\sigma_1)/2} \le \frac{cb}{2(1-c\Omega(\sigma_1)/2)} \equiv e_0 \tag{3.37}$$

holds for all $\varepsilon \in S(0,\gamma;E)$ and $\xi \in D_{\sigma_1}(0)$. For $i \ge 1$, we consider the modification of Nagumo norms:

$$\|y\|_j = \sup_{\xi \in D_{\sigma_1}(0)} (d_{\sigma_1}(\xi))^j \sum_{n=0}^{\infty} \sup_{\varepsilon \in S(0,\gamma;E)} \left| \frac{1}{n!}\frac{\partial^n y}{\partial \xi^n}(\varepsilon,0) \right| |\xi|^n,$$

with $d_{\sigma_1}(\xi) = \sigma_1 - |\xi|$. It follows from (3.36) that

$$\|\phi_i\|_{i-1} \le \frac{c}{1-c\Omega(\sigma_1)} \|G_i\|_{i-1}, \tag{3.38}$$

where, by (3.26), (3.11), (3.9) and Hypothesis 1.1,

$$\begin{aligned} \|G_1\|_0 &= \sup_{\xi \in D_{\sigma_1}(0)} \sum_{n=1}^{\infty} n \sup_{\varepsilon \in S(0,\gamma;E)} |a_n(\varepsilon)|\,|\xi|^n + \sup_{\varepsilon \in S(0,\gamma;E)} \frac{|\beta_1(\varepsilon)|}{2} \\ &< \frac{b\sigma_1}{2(\kappa-\sigma_1)} + \frac{bm}{2}\,, \end{aligned}$$

and by (3.27), together with the properties of Nagumo norms,

$$\|G_i\|_{i-1} \le \|\xi\|_0\, \|\phi'_{i-1}\|_{i-1} + 2\|\xi\|_0 \sum_{j=1}^{i-1} \|\phi_j\|_{j-1}\, \|\phi_{i-j}\|_{i-j} + \frac{\|\beta_i\|_{i-1}}{2}\,, \qquad i \ge 2. \tag{3.39}$$

From this, together with (3.38), a recursive relation of the same type studied in **(b)** may be derived for the $\|\phi_l\|_{l-1}$ (see (3.18)-(3.21)) and one may conclude that[7]

$$\|\phi_l\|_{l-1} \leq \Delta \frac{Al!^{s'}}{l^2} \omega^l \tag{3.40}$$

holds for all $l \geq 1$ and some suitable constants Δ and ω. Picking $\sigma < \sigma_1$ in the property 5 of Nagumo norms, yields

$$|\phi_i(\varepsilon, \xi)| \leq \frac{\sigma_1}{(\sigma_1 - \sigma)^i} \|\phi_i\|_{i-1} \leq Ci!^{s'} \mu^i$$

for all $i \geq 1$ uniformly in $S(0, \gamma; E) \times \bar{D}_\sigma(0)$, with $C = \sigma_1 \Delta A$ and $\mu = \omega/(\sigma_1 - \sigma)$. We choose $C = \max(e_0, \sigma_1 \Delta A)$ in order to include the $i = 0$ case. This concludes the proof of Lemma 3.4.

□

3.5 Concluding this Section

Theorem 3.7 *Let $\beta = \beta(\varepsilon)$ obey Hypothesis 1.1 and let (3.3) be considered for (ε, ξ) in a domain $S(0, \gamma; E) \times D_\sigma(0)$ with $\gamma > \pi/s'$. Then, there exist a radius $\sigma > 0$ such that for $\xi \in \bar{D}_\sigma(0)$ the formal solution $\hat{y}(\varepsilon, \xi)$ is s'-summable in $\theta = 0$ direction.*

Proof By Taylor's Theorem

$$r_I(\varepsilon, \xi) = \varepsilon^{-I} \left(y(\varepsilon, \xi) - \sum_{i=0}^{I-1} y_i(\xi)\, \varepsilon^i \right) = \frac{I}{\varepsilon^I} \int_0^\varepsilon y_I(\xi, \xi)\, (\varepsilon - \xi)^{I-1}\, d\xi,$$

where the integral is along a path from 0 to ε inside $S(0, \gamma; E)$. This, together with Lemma 3.4, implies

$$|r_I(\varepsilon, \xi)| \leq CI!^{s'} \mu^I$$

for every I and $(\varepsilon, \xi) \in S' \times \bar{D}_\sigma(0)$, with S' any proper subsector of $S(0, \gamma; E)$. In addition, Lemma 3.3 states that $\hat{y}(\varepsilon, \xi)$, a formal solution of (3.3), is an element of $\mathcal{O}(\sigma)[[\varepsilon]]_s$; therefore is an element of $\mathcal{O}(\sigma)[[\varepsilon]]_s$ for any σ. Take now $\sigma < \kappa/(1+2bc\kappa)$ with b, c and κ satisfying (3.9), (3.8) and (3.10), respectively. Hence, by Definition 2.1, $\hat{y}(\varepsilon, \xi)$ is an asymptotic expansion of order s', as $\varepsilon \to 0$ in the sector $S(0, \gamma; E)$, of $y(\varepsilon, \xi)$, which by Lemma 3.2 is an analytic solution of (3.3) in the domain $S(0, \gamma; E) \times \bar{D}_\sigma(0)$. Then, as $\gamma > \pi/s'$, by hypothesis, $y(\varepsilon, \xi)$ is the only Gevrey order s' asymptotic expandable function in $S(0, \gamma; E)$ which has $\hat{y}(\varepsilon, \xi)$ as its asymptotic expansion, and $\hat{y}(\varepsilon, \xi)$ is s'-summable in $\theta = 0$ direction by Definition 2.4.

□

[7]The details for this estimate are left to the reader.

4 The Evolution Equation

We derive, at the level of formal manipulations, an equation for the Legendre transform $w(t,p)$ of $u(t,x)$ (see (1.9), for definition) which is suitable for generating a formal power series in $\varepsilon = 2/N$, which belongs to $\bigcap_{t\geq 0} \mathcal{O}(r(t))[[\varepsilon]]$, for its derivative $v(t,p) = w_p(t,p)$:

$$\hat{v}(t,\varepsilon,p) = \sum_{i=0}^{\infty} v^i(t,p)\varepsilon^i \ , \tag{4.1}$$

in the sense that its i–th coefficient $v^i(t,p)$ satisfies a solvable first order linear PDE equation, with an external source $f_i = f_i(v^0,\ldots,v^{i-1})$ depending on the coefficients $v^j(t,p)$ with $j < i$.

For $i = 0$, $f_0 = -1$, the equation for $v^0(t,p)$ can be integrated and the solution can be written in terms of ordinary analytic functions (see (4.12)). For any $i > 1$ it can be shown that its solution is a uniquely defined holomophic function of p for every $t \geq 0$. The two results presented in this Section are extensions of the **(a)** and **(b)** statements for $t > 0$. Theorem 4.3 states that the coefficients $(v^i(t,p))_{i\geq 0}$ are analytic functions in a monotone non-increasing domains $D_i \equiv D_{\sigma_i}(-1)$, uniformly in $t \geq 0$ and Theorem 4.5 finds a majorant solution at the critical point; Theorem 4.6 states that the domains D_i may be chosen in such way that $\lim_{i\to\infty}\sigma_i = \varsigma > 0$ and the formal series (4.1) is of Gevrey order 2, uniformly in $t \in \mathbb{R}_+$.

The functions $v(t,\varepsilon,p)$ and $u_x(t,\varepsilon,x)$ are inverse of each other and in a separate paper [CM, CM] we prove that, in general, (i) if a formal series $\hat{f}(\varepsilon,z)$ is in $\mathcal{O}(r)[[\varepsilon]]$, then the formal series $\hat{g}(\varepsilon,w)$ for its inverse is in $\mathcal{O}(\rho)[[\varepsilon]]$ for some $\rho > 0$; (ii) if $\hat{f}(\varepsilon,z) \in \mathcal{O}(r)[[\varepsilon]]_s$ is, in addition, a Gevrey asymptotic expansion of a holomorphic function $f(\varepsilon,z)$, then $\hat{g}(\varepsilon,w) \in \mathcal{O}(\rho)[[\varepsilon]]_s$ is a Gevrey asymptotic expansion of the holomorphic function $g(\varepsilon,w) = f^{-1}(\varepsilon,w)$. Moreover, $\hat{f}(\varepsilon,z)$ is s–summable if, and only if, $\hat{g}(\varepsilon,w)$ is s–summable. So, we may work in one or another side depending on the convenience. In this Section, the series (4.1) turns out to be more convenient to work with than the formal series $\hat{u}_x(t,x)$. The theorems just mentioned guarantee that any asymptotic property for the formal series (4.1) in ε is preserved by the inverse operation in the variable p, uniformly in $t > 0$.

4.1 Equation for the Legendre Transform

We assume that the Legendre transform (1.9) of $u(t,x)$, the solution of equation (1.7), is attained at a unique solution $\bar{x} = \bar{x}(t,p)$ of $p = u_x(t,x)$, for every $t \geq 0$ and p in a certain domain depending on $\varepsilon = 2/N$ and t, and the function $u(t,x)$ can be recovered by the inverse Legendre transform:

$$u(t,x) = \max_p (xp - w(t,p)) = x\bar{p} - w(t,\bar{p}), \tag{4.2}$$

where $\bar{p} = \bar{p}(t, x)$ is the unique solution of $x = w_p(t, p)$. Substituting

$$\begin{aligned} u_t &= -w_t \\ x &= w_p \\ u_x &= p \\ u_{xx} &= 1/w_{pp} \end{aligned} \tag{4.3}$$

into (1.7), we have

$$w_t = -\varepsilon \frac{w_p}{w_{pp}} - p + 2p^2 w_p + (d+2)pw_p - dw + p_0(t)$$

where $p_0(t)$ is the solution of $w_p(t, p) = 0$. Taking derivative with respect to x, we arrive at the following initial value problem for $v = w_p$:

$$v_t - 2p(1+p)v_p = (d+2+4p)v - 1 - \varepsilon \left(1 - \frac{vv_{pp}}{(v_p)^2}\right) \tag{4.4}$$

with $v(0, p) = (u_0')^{-1}(p) \equiv v_0(p)$ where $u_0(x)$ is given by (1.8).

The formal series (4.1) at $t = 0$ has to agree with

$$\hat{v}_0(\varepsilon, p) = \sum_{i=0}^{\infty} v_0^i(p)\varepsilon^i \ . \tag{4.5}$$

In order to generate a formal series for $v_0(p)$, it will be convenient to write an equation for the initial condition of (4.4) as well. Using (4.3) into (3.3) ($y = u_x = p$, $y' = u_{xx} = 1/v_0'$ and $\xi = x = v_0$), we have

$$\varepsilon \frac{v_0(p)}{v_0'(p)} + p - 2p^2 v_0(p) + \frac{\beta}{2} = 0 \tag{4.6}$$

with $\beta = \beta(\varepsilon)$ obeying Hypothesis 1.1. Taking $\varepsilon = 0$ in (4.6), yields

$$v_0^0(p) = \frac{2p + \beta^0}{4p^2} \ . \tag{4.7}$$

4.1.1 Power Series in the Variable ε

Without loss of generality, we shall restrict ourselves to the case $d = 4$. The 0–th coefficient of the expansion (4.1) has been treated in [MCG] with considerable details for dimension $d = 4$ and this explain our choice. Similar results can be obtained for any $d > 2$.

Substituting the formal power series (4.1) into (4.4) (with $d = 4$), collecting all terms of the same order, we obtain the following first order linear partial differential equation satisfied by the coefficient $v^i(t, p)$ with $i \geq 0$:

$$v_t^i - 2p(1+p)v_p^i = (6+4p)v^i + f_i(v^i, \ldots, v^i) \ , \tag{4.8}$$

with initial value $v^i(0,p) = v_0^i(p)$, where $f_0 = -1$;

$$\begin{aligned} f_1(v^0) &= -1 + \frac{v^0 v^0_{pp}}{\left(v^0_p\right)^2} \\ &= -1 + v^0 \varphi^0_p \end{aligned} \tag{4.9}$$

with $\varphi^0 = -1/v_p^0$; and for $i \geq 2$,

$$f_i(v^0, \ldots, v^{i-1}) = \sum_{j=0}^{i-1} v^{i-1-j}\, \varphi^j_p; \tag{4.10}$$

where φ^j_p is the derivative of φ^j with respect to p and φ^j , $j \geq 1$, is obtained by Scott's formula (2.8), applied formally to φ^0:

$$\begin{aligned} \varphi^j &= \frac{1}{j!} \frac{\partial^j}{\partial \varepsilon^j} \frac{-1}{v_p} \bigg|_{\varepsilon=0} \\ &= \sum_{k=1}^{j} \frac{(-1)^{k+1}}{\left(v^0_p\right)^{k+1}} \sum_{\substack{l_1,\ldots,l_k \geq 1: \\ l_1+\cdots+l_k=j}} v_p^{l_1} \cdots v_p^{l_k} \ . \end{aligned} \tag{4.11}$$

The solution for $i = 0$, reads (see Theorem 2.2 of [MCG])

$$v^0(t,p) = \frac{2+p}{2p^2} - \frac{4-\beta^0}{4p^2} e^{2t} - \frac{1+p}{p^3}\left[2t + \ln\left(1 - (1-e^{-2t})(1+p)\right)\right] \ , \tag{4.12}$$

which is a holomorphic function of p in $D_1(-1)$ for every $t \in \mathbb{R}_+$. For $\beta^0 = \beta^0_c(d = 4) = 4$, the exponentially growing term is cancelled and its derivative $v^0_p(t,p)$ vanishes at $p = p^*(t) \in \mathbb{R}$, which is the point with least real coordinate in the domain $\Omega_t = u_x(t, \mathbb{H})$ referred in the Introduction. So, from what we have explained there and according to Figure , $p^*(t)$ is a monotone increasing function of $t \geq 0$ with $p^*(0) = -4$ and $p^*(\infty) = -3/2$. Therefore, $1/v^0_p(t,p)$ is holomorphic for $(t,p) \in \{\mathbb{R}_+ \times \mathbb{C} : |1+p| < |1+p^*(t)|\}$ and particularly for $(t,p) \in \mathbb{R}_+ \times D_{1/2}(-1)$.

As a consequence, supposing that each $v^j(t,p)$, for $j = 0, 1, \ldots, i-1$, is holomorphic in $(t,p) \in \mathbb{R}_+ \times D_{\sigma_j}(-1)$, it follows by (4.10) and (4.11) that $f_i(v^0, \ldots, v^{i-1})$ is holomorphic for $t \in \mathbb{R}_+$ and p in $\bigcap_{j=0}^{i-1} D_{\sigma_j}(-1) \cap D_{1/2}(-1)$. Without loss of generality, it is enough to take monotone non-increasing sequences $(\sigma_i)_{i \geq 0}$:

$$\cdots \leq \sigma_i \leq \sigma_{i-1} \leq \cdots \leq \sigma_1 \leq \sigma_0 = 1/2 \ . \tag{4.13}$$

Holomorphic Domain of the Initial Condition

Proposition 4.1 *The i–th coefficient $v_0^i(p)$ of the formal series (4.5) is a holomorphic function in $D_1(-1)$ for all $i \geq 0$. For each $i \geq 0$, the value of $v_0^i(p)$ at $p = -1$ depends on the value of $v_0^j(p)$, for $j = 0, \ldots, i-1$, at $p = -1$ and on an arbitrary real number β^i defined by (1.12) in such way that, if $\beta^0, \ldots, \beta^{i-1}$ are fixed, $v_0^i(-1) = a_i + \beta^i/4$ for some $a_i \in \mathbb{R}$ fixed.*

Remark 4.2 *It follows by Proposition 4.1 that the formal power series $\hat{\beta}(\varepsilon) = \sum_{i \geq 0} \beta^i \varepsilon^i$ of the inverse temperature is of Gevrey order of the formal power series (4.5). By Lemma 3.3, the formal series $\hat{y}(\varepsilon, \xi) = \hat{u}_0'(\varepsilon, \xi) = \hat{v}_0^{-1}(\varepsilon, \xi)$ for the inverse of $\hat{v}_0(\varepsilon, p)$ is of Gevrey order $s' = \min(s, 1)$ where s is the Gevrey order of $\hat{\beta}(\varepsilon)$, by Hypothesis 1.1. Since the Gevrey order is preserved by the inverse function theorem (see Theorem 3.3 of [CM1]) we conclude that $s' = s = 1$ and $\hat{\beta}(\varepsilon)$ is of Gevrey order* 1.

Proof. The coefficients of (4.5) can be determined explicitly from the equation (4.6). Plugging the formal power series (4.5) into (4.6), taking into account (1.12) and collecting all terms of the same order, yields

$$v_0^i(p) = \frac{1}{4p^2}\left(g_i(v_0^0, \ldots, v_0^{i-1}) + \beta^i\right), \tag{4.14}$$

where $g_0 = 2p$ and for $i \geq 1$

$$g_i(v_0^0, \ldots, v_0^{i-1}) = -2\sum_{j=0}^{i-1} v_0^{i-1-j}\, \varphi_0^j \tag{4.15}$$

with $\varphi_0^j = \varphi^j|_{t=0}$, φ^j given by (4.11):

$$\varphi_0^0 = \frac{-1}{dv_0^0/dp}$$

and for $j \geq 1$

$$\varphi_0^j = \sum_{k=1}^{j} \frac{(-1)^{k+1}}{\left(dv_0^0/dp\right)^{k+1}} \sum_{\substack{l_1, \ldots, l_k \geq 1: \\ l_1 + \cdots + l_k = j}} \frac{dv_0^{l_1}}{dp} \cdots \frac{dv_0^{l_k}}{dp}\,.$$

Choosing $\beta^0 = \beta_c^0(4) = 4$ in (4.7), $v_0^0(p) = (2+p)/2p^2$ is a holomorphic function for p in the disc centered at $p = -1$ of radius 1 and

$$\frac{dv_0^0}{dp}(p) = -\frac{4+p}{2p^3}$$

vanishes only at $p = -4$. Since the space of a meromorphic functions of p, with poles at $p = -4$ and $p = 0$, are closed by differentiation, multiplication and linear combinations with constant coefficients, $v_0^i(p)$ is a meromorphic function of p with poles at $p = -4$ and $p = 0$ and, consequently, holomorphic in $D_1(-1)$ for all $i \geq 1$.

The statement on $v_0^i(-1)$ can be read from (4.14). Note that $v_0^i(p)$ is a real analytic function by (4.14) and the Hypothesis 1.1. For $\beta^0, \ldots, \beta^{i-1}$ fixed, $g_i(v_0^0, \ldots, v_0^{i-1})/4p^2$ at $p = -1$ is a fixed real number, let say a_i and $v_0^i(-1) = a_i + \beta^i/4$.

$\square$

Later on, we shall need also a lower bound for dv_0^0/dp in $D_{\sigma_0}(-1)$. Maximum (minimum) modulus principle will be used for this purpose in Subsection 4.2.

Holomorphic Domain of the Initial Value Problem We are now ready to stated the main result of this subsection. From here on, $d = 4$ and $\beta = \beta(\varepsilon)$ obeys Hypothesis 1.1. We shall establish that $v^i(t,p)$ is holomorphic in $D_{\sigma_i}(-1)$ for $t > 0$ and $i \geq 0$. Furthermore, we shall obtain in each of these domains precise upper bounds, with explicit t dependence, for $v^i(t,p)$ and its derivatives with respect to p. This allows us to find the critical value β_c^i for each initial value (4.8), $i \geq 0$, and determine the formal series $\hat{\beta}_c = \sum_{i\geq 0} \beta_c^i \varepsilon^i$ for the critical inverse temperature. We shall fix the coefficients β^i of the formal power series $\hat{\beta}(\varepsilon)$, introduced in the Hypothesis 1.1 and whose dependence on the initial conditions $v_0^i(p)$ is given by (4.14), on their critical values. Only β^0 needs to be fixed at its critical value $\beta_c^0 = 4$ for the first part of the following

Theorem 4.3 *The initial value problem (4.4), satisfied by each coefficient $v^i(t,p)$ of the formal power series (4.1), has an unique solution given by*

$$v^i(t,p) = \frac{1}{\alpha(p)} \left((\alpha v_0^i) \circ \phi_t(p) + \int_0^t (\alpha f_i) \circ \phi_{t-s}(p) \, ds \right), \tag{4.16}$$

holomorphic in $D_{\sigma_i}(-1)$ with σ_i satisfying (4.13), uniformly in $t > 0$. Here,

$$\alpha(p) = \frac{2p^3}{1+p} \tag{4.17}$$

is related to an integral factor $\mu = \mu(p)$: $\alpha = 2p(1+p)\mu$ and

$$\phi_t(p) = \frac{pe^{2t}}{1+p-pe^{2t}} \tag{4.18}$$

solves the characteristic of equation (4.8).

Proof The proof of Theorem 4.3 has two steps.

First step. Assume that, for each $i \geq 1$, $f_i(v^0, \ldots, v^{i-1})$ given by (4.10) and (4.11), is holomorphic in $D_{\sigma_{i-1}}(-1)$ for $t \in \mathbb{R}_+$, so $\alpha f_i(v^0, \ldots, v^{i-1})$ has a simple pole at $p = -1$ and can be written as a Laurent series

$$\alpha f_i(t,p) = \frac{\gamma_{-1}^i(t)}{1+p} + \sum_{n=0}^{\infty} \gamma_n^i(t) \, (1+p)^n \tag{4.19}$$

with the series in the right hand side of (4.19) being convergent for $p \in D_{\sigma_{i-1}}(-1)$, uniformly in $t > 0$. As v_0^i is holomorphic in $D_1(-1)$, we analogously have

$$\alpha v_0^i(p) = \frac{c_{0,-1}^i}{1+p} + \sum_{n=0}^{\infty} c_{0,n}^i \, (1+p)^n \tag{4.20}$$

for $p \in D_1(-1)\backslash\{-1\}$.

Write

$$\vartheta^i(t,p) := \alpha(p) v^i(t,p) = \frac{c_{-1}^i(t)}{1+p} + \sum_{n=0}^{\infty} c_n^i(t) \, (1+p)^n . \tag{4.21}$$

In terms of this new function, the problem (4.4), which reads

$$\vartheta_t^i - 2p(1+p)\vartheta_p^i = \alpha f_i(t,p) \tag{4.22}$$

with $\vartheta^i(0,p) = \vartheta_0^i(p) = \alpha v_0^i(p)$, can be solved by the *methods of characteristics* (see e.g. [E]). $\vartheta^i(t,p)$ on the characteristics $p = p(t)$:

$$V(t) = \vartheta^i(t, p(t)),$$

satisfies $\dot V = \vartheta_t^i + \dot p \vartheta_p^i$ which, by comparing with (4.22), reduces the inicial value problem to a system of ordinary differential equations

$$\begin{array}{rcl} \dot p &=& -2p(1+p) \\ \dot V &=& \alpha f_i(t,p) \ , \end{array} \tag{4.23}$$

with $p(0) = p_0$ and $V(0) = \vartheta^i(0, p(0)) = \vartheta_0^i(p_0)$. Integrating the first equation of (4.23)

$$\int_{p_0}^{p} \frac{dp'}{p'(1+p')} = \int_{p_0}^{p} \left(\frac{1}{p'} - \frac{1}{1+p'}\right) dp' = -2 \int_0^t dt',$$

we are lead to

$$p(t) = p(t, p_0) = \phi_{-t}(p_0) \tag{4.24}$$

where $\phi_t(p)$ is given by (4.18). The second equation of (4.23) may also be integrated:

$$V(t) = \vartheta_0^i(p_0) V(0) + \int_0^t \alpha f_i(s, p(s,p_0)) \, ds \ . \tag{4.25}$$

According to the methods of characteristics, the solution $\vartheta^{[i]}(t,p)$ is recovered by substituting p_0, as a function of t and p, into (4.25). Solving (4.24) for p_0, gives

$$p_0(t,p) = \phi_t(p) \, ;$$

note that $p(s,p_0) = \phi_s^{-1}(p_0) = \phi_{-s}(p_0)$ and $\phi_{-s} \circ \phi_t(p) = \phi_{t-s}(p)$. Plugging this expression into (4.25), yields

$$\vartheta^i(t,p) = \vartheta_0^i \circ \phi_t(p) + \int_0^t (\alpha f_i) \circ \phi_{t-s}(p) \, ds \tag{4.26}$$

which gives (4.16), by the definition (4.21) of $\vartheta^{[i]}(t,p)$.

It follows from (4.18) that $1+\phi_t(p) = (1+p)\,/\,(1+p-pe^{2t})$ is an analytic function of p for $(t,p) \in \mathbb{R}_+ \times D_1(-1)$, satisfies $1+\phi_t(-1) = 0$ and

$$\max_{p\in D_\sigma(-1)} |1+\phi_t(p)| < \frac{\sigma e^{-2t}}{1-\sigma(1-e^{-2t})} \leq \sigma \tag{4.27}$$

for every $\sigma \leq 1$ and all $t \geq 0$. This, together with (4.19), (4.20) and (4.21), implies that $v^i(t,p) = \vartheta^i(t,p)/\alpha(p)$ is holomorphic in $D_{\sigma_i}(-1)$ with $\sigma_i = \min(1,\sigma_{i-1})$ and concludes the proof of Theorem 4.3 under the hypothesis (4.19).

Second step. We shall prove that $v^i(t,p)$ is holomorphic in $D_{\sigma_i}(-1)$, for $i \geq 0$ and $t \geq 0$. We recall that β^0 is kept fixed at its critical values β_c^0: $\beta^0 = 4$. We have seen that, under this condition, for every $t \geq 0$, $v^0(t,p)$ is holomorphic in $D_1(-1)$ and its derivative $v_p^0(t,p)$ does not vanishes in $D_{1/2}(-1)$. By (4.9), f_1 is holomorphic in $D_{1/2}(-1)$ and by (4.26), $v^1(t,p) = \vartheta^1(t,p)/\alpha(p)$ is holomorphic in $D_{\sigma_1}(-1)$ with $\sigma_1 \leq \min(1,1/2) = 1/2$, proving the theorem statement for $i=1$. We now assume that each $v^j(t,p)$, for $j=1,\ldots,i-1$, is holomorphic in $(t,p) \in \mathbb{R}_+ \times D_{\sigma_j}(-1)$. Since $f_i(v^0,\ldots,v^{i-1})$ is holomorphic in $\mathbb{R}_+ \times D_{\sigma_{i-1}}(-1)$ (a finite sum of products of holomorphic functions) it follows by (4.26) that $v^i(t,p) = \vartheta^i(t,p)/\alpha(p)$ is holomorphic in $\mathbb{R}_+ \times D_{\sigma_i}(-1)$ with $\sigma_i \leq \min(1,\sigma_{i-1}) = \sigma_{i-1}$. This completes the proof of Theorem 4.3.

$\square$

We are now concerned with the dependence on t of $\gamma_{-1}^i(t)$ and $\gamma_n^i(t)$, $n \geq 0$. We are going to show that these quantities remain bounded in t for every $i \geq 1$.

Proposition 4.4

$$\begin{aligned} \left|\gamma_{-1}^i(t)\right| &\leq \delta_i \\ \left|\gamma_n^i(t)\right| &\leq \Delta_i \kappa_i^{-n}\,, \qquad n \geq 0, \end{aligned} \tag{4.28}$$

hold for some $0 < \delta_i,\ \Delta_i < \infty$, $\kappa_i < \sigma_i$ *and* $i \geq 1$, *uniformly in* $t \geq 0$.

Proof Proposition 4.4 will be proven by induction. The dependence on t of f_1 can be explicitly evaluated from the solution (4.12). We write

$$v^0(t,p) = \frac{-4t}{\alpha(p)} + H_0(t,p) \tag{4.29}$$

and note that $H_0(t,p)$ and its first and second derivatives remain bounded uniformly in $\mathbb{R}_+ \times \bar{D}_{\kappa_1}(-1)$, with $\kappa_1 < \sigma_1$. Observe that the numerator $v^0 v_{pp}^0 - \left(v_p^0\right)^2$ and the denominator $\left(v_p^0\right)^2$ of (4.9), both increase as t^2 and we have

$$\begin{aligned} \left|\gamma_{-1}^1(t)\right| &\leq \delta_1 \\ \left|\gamma_n^1(t)\right| &\leq \Delta_1 \kappa_1^{-n}\,, \qquad n \geq 0, \end{aligned}$$

for some positive and finite constants δ_1 and Δ_1, uniformly in $t \geq 0$. Bounds for $n \geq -1$ are obtained by Cauchy formula $\gamma_n^1(t) = \dfrac{1}{2\pi}\displaystyle\int_{|z|=\kappa_1} \alpha(z-1) f_1(v^0(t, z-1))/z^{n+1} dz$.

Now, assume the inequalities (4.28) hold for $i = 1, \ldots, k$, uniformly in $t \geq 0$. We shall prove that these hold for $i = k+1$. We shall first majorize the solution of (4.22), under the assumption (4.28).

4.1.2 Power Series Expansion Around $p = -1$

Plugging

$$\frac{1}{1+\phi_t(p)} = \frac{e^{2t}}{1+p} + 1 - e^{2t}$$

and

$$\begin{aligned}
(1+\phi_t(p))^n &= e^{-2t} \sum_{j_1=0}^{\infty} (1-e^{-2t})^{j_1} (1+p)^{j_1+1} \cdots e^{-2t} \sum_{j_n=0}^{\infty} (1-e^{-2t})^{j_n} (1+p)^{j_n+1} \\
&= e^{-2nt} \sum_{j=0}^{\infty} \binom{n+j-1}{j-1} (1-e^{-2t})^j (1+p)^{n+j} \qquad (4.30)
\end{aligned}$$

for all $n \geq 1$, into (4.26) with αv_0^i and αf_i given respectively by (4.19) and (4.20), we have for $p \in D_{\sigma_i}(-1) \setminus \{-1\}$

$$\begin{aligned}
\vartheta^i(t,p) &= c_{0,-1}^i e^{2t} \frac{1}{1+p} + c_{0,-1}^i (1-e^{2t}) + c_{0,0}^i + \sum_{n=1}^{\infty} C_n^i(t) (1+p)^n \\
&\quad + \int_0^t \left(\gamma_{-1}^i(s) e^{2(t-s)} \frac{1}{1+p} + \gamma_{-1}^i(s) \left(1 - e^{2(t-s)}\right) \right. \\
&\quad \left. + \gamma_0^i(s) + \sum_{n=1}^{\infty} \Gamma_n^i(s,t) (1+p)^n \right) ds \ , \qquad (4.31)
\end{aligned}$$

with

$$\left\{ \begin{array}{l} C_n^i(t) \\ \Gamma_n^i(t,t') \end{array} \right. = (1-e^{-2t})^n \sum_{k=1}^{n} \left\{ \begin{array}{l} c_{0,k}^i \\ \gamma_k^i(t'-t) \end{array} \right. \binom{n-1}{k-1} \left(\frac{e^{-2t}}{1-e^{-2t}} \right)^k .$$

Note that, under the hypothesis (4.28), we have

$$\begin{aligned}
\left|\Gamma_n^i(t,t')\right| &\leq \Delta_i (1-e^{-2t})^n \sum_{k=1}^{n} \binom{n-1}{k-1} \left(\frac{e^{-2t}}{\sigma_i(1-e^{-2t})} \right)^k \\
&= \Delta_i \frac{e^{-2t}(\sigma_i + (1-\sigma_i)e^{-2t})^{n-1}}{\sigma_i^n} \qquad (4.32)
\end{aligned}$$

and, similarly

$$\left|C_n^i(t)\right| \leq c_i \frac{e^{-2t}(\sigma + (1-\sigma)e^{-2t})^{n-1}}{\sigma^n} \tag{4.33}$$

since, by Cauchy formula, $|c_n^i| \leq c_i \sigma^{-n}$ holds for any $\sigma < 1$ for some constant $0 < c_i < \infty$.

The Critical Value We observe that the residue and the constant coefficient of the Laurent series of ϑ^i about $p = -1$ increase exponentially fast with t. Such growth must be cancelled by setting the j–th coefficient β^i of the formal power series $\hat{\beta}(\varepsilon)$ for the inverse temperature at its critical value. Assuming that we have already fixed $\beta^j = \beta_c^j$ for $j = 0, 1, \ldots, i-1$, β_c^i is chosen so that

$$\lim_{t\to\infty} e^{-2t} c_{-1}^i(t) = \lim_{t\to\infty} e^{-2t} \left.(1+p)\vartheta^i(t,p)\right|_{p=-1} = 0$$

or, equivalently,

$$\Lambda^i := c_{0,-1}^i + \int_0^\infty \gamma_{-1}^i(s) e^{-2s}\, ds = 0 \ , \tag{4.34}$$

by (4.31), Note that the exponential growth of the constant coefficient in (4.31) is eliminated by the very same condition:

$$\lim_{t\to\infty} e^{-2t} c_0^i(t) = \lim_{t\to\infty} e^{-2t} \left[\vartheta^i(t,p) - \left.(1+p)\vartheta^i(t,p)\right|_{p=-1} \frac{1}{1+p}\right]_{p=-1} = 0$$

if and only if (4.34) holds. By (4.21), (4.17) and (4.14), both $c_{0,-1}^i$ and $c_{0,0}^i$ depend on β^i. All other coefficients depends on the β^j with $j < i$, which we have fixed before (see Proposition 4.1, for more explicit information). We shall denote by $\bar{c}_{0,-1}^i = c_{0,-1}^i(\beta_c^i)$ and $\bar{c}_{0,0}^i = c_{0,0}^i(\beta_c^i)$ the former two constants for β^i at its critical value β_c^i.

4.1.3 Cauchy Majorant Method

We give an alternative prove of the existence and uniqueness of the solution of (4.8), holomorphic in $D_{\sigma_i}(-1)$, by Cauchy majorant method (see e.g. [Fo]). A subproduct of this method is the factorization of a linearly increases in t term from $v^i(t,p)$, for $i \geq 1$, similar to equation (4.29) for $i = 0$:

Theorem 4.5 *Assume that β_i has been fixed at its critical value, according to (4.34), for every $i \geq 1$ for which (4.28) hold. Then*

$$v^i(t,p) \ll \frac{-\eta_i}{\alpha(p)} t + H_i(t,p)$$

for $0 < \eta_i = \delta_i + \Delta_i < \infty$ and a holomorphic function $H_i(t,p)$ of p in $D_{\sigma_i}(-1)$, bounded in any closed domain $\bar{D}_{\kappa_i}(-1)$ with $\kappa_i < \sigma_i$, uniformly in t.

Proof The collection of terms with exponentially increasing factor in (4.31) can be written as

$$e^{2t}\left(c^i_{0,-1}+\int_0^t \gamma^i_{-1}(s)e^{-2s}\,ds\right)\left(\frac{1}{1+p}-1\right)=e^{2t}\left(-\Lambda^i+\int_t^\infty \gamma^i_{-1}(s)e^{-2s}\,ds\right)\frac{p}{1+p}$$

with $\Lambda^i=0$ at the critical value. We observe that the integral in the complementary interval $[t,\infty)=\mathbb{R}_+\backslash[0,t)$, remains bounded under the hypothesis (4.28):

$$\left|e^{2t}\int_t^\infty \gamma^i_{-1}(s)e^{-2s}\,ds\right|\leq e^{2t}\int_t^\infty \left|\gamma^i_{-1}(s)\right|e^{-2s}\,ds\leq\frac{\delta_i}{2}\ . \tag{4.35}$$

The same is true for both sums in (4.31) for (t,p) in compact subsets of $\mathbb{R}_+\times D_{\sigma_i}(-1)$. For this, the following integration

$$\int_0^t e^{-2s}(\sigma+(1-\sigma)e^{-2s})^{n-1}ds=\frac{1}{2n}\frac{1}{1-\sigma}\left(1-(\sigma+(1-\sigma)e^{-2t})^n\right)$$

results by changing to variable $y=e^{-2s}$. By (4.33) and (4.32), we have

$$\begin{aligned}\sum_{n=1}^\infty C^i_n(t)\,(1+p)^n &\ll c_ie^{-2t}\sum_{n=1}^\infty\frac{(\sigma+(1-\sigma)e^{-2t})^{n-1}}{\sigma^n}(1+p)^n\\ &= c_i\sigma^{-1}e^{-2t}\frac{1+p}{1-(1+(\sigma^{-1}-1)e^{-2t})\,(1+p)}\equiv F^i(t,p)\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}\int_0^t\sum_{n=1}^\infty\Gamma^i_n(s,t)\,(1+p)^nds &\ll \frac{\Delta_i}{1-\sigma_i}\sum_{n=1}^\infty\frac{1}{2n}\frac{(1+p)^n}{\sigma_i^n}\left(1-(\sigma_i+(1-\sigma_i)e^{-2t})^n\right)\\ &= \frac{\Delta_i}{2(1-\sigma_i)}\ln\frac{1-(1+(\sigma_i^{-1}-1)e^{-2t})\,(1+p)}{1-\sigma_i^{-1}\,(1+p)}\\ &\equiv G^i(t,p)\end{aligned} \tag{4.37}$$

and these functions are bounded in every closed domain $\bar{D}_{\kappa_i}(-1)$ with $\kappa_i<\sigma_i$, uniformly in $t\in\mathbb{R}_+$.

Only the constant coefficient in (4.31) increases (linearly) with t:

$$\left|\int_0^t\left(\gamma^i_{-1}(s)+\gamma^i_0(s)\right)ds\right|\leq\int_0^t\left(\left|\gamma^i_{-1}(s)\right|+\left|\gamma^i_0(s)\right|\right)ds\leq(\delta_i+\Delta_i)t\ .$$

Puting all together, yields

$$(1+p)\vartheta^i(t,p)\ll\frac{\delta_i}{2}(1+\sigma_i)+\left(\left|\bar{c}^i_{0,-1}\right|+\left|\bar{c}^i_{0,0}\right|+F^i(t,p)+(\delta_i+\Delta_i)t+G^i(t,p)\right)(1+p)$$

with $c^i_{0,-1}=c^i_{0,-1}(\beta^i)$ and $c^i_{0,0}=c^i_{0,0}(\beta^i)$ evaluated at the critical value β^i_c.

By (4.21),

$$v^i(t,p) = \frac{1}{\alpha(p)}\vartheta^i(t,p) = \frac{1+p}{2p^3}\sum_{n=-1}^{\infty} c_n^i(t)\,(1+p)^n = \sum_{n=0}^{\infty} a_n^i(t)\,(1+p)^n \tag{4.38}$$

is majorized ($|a_n^i(t)| \le A_n^i(t)$)

$$v^i(t,p) \ll \frac{-\eta_i}{\alpha(p)}t + H^i(t,p) \tag{4.39}$$

with $\eta_i = \delta_i + \Delta_i$, $A_n^i(t) = (n+1)(n+2)\eta_i/4 + (1/n!)\partial^n H^i/\partial p^n(t,-1)$ and

$$H^i(t,p) = \frac{\delta_i(1+\sigma_i)}{4(1-\sigma_i)^3} - \frac{1}{\alpha(p)}\left(\left|\bar{c}^i_{0,-1}\right| + \left|\bar{c}^i_{0,0}\right| + F^i(t,p) + G^i(t,p)\right) \tag{4.40}$$

bounded in any closed domain $\bar{D}_{\kappa_i}(-1)$ with $\kappa_i < \sigma_i$, uniformly in t. Note that $-1/\alpha(p)$ has power series in $1+p$ with positive coefficients.

□

Proof of Proposition 4.4 – ***Completion*** We shall complete the induction step. By hypothesis, (4.28) holds for each $i = 1, \ldots, k$ and $v^i(t,p)$ satisfies a majorization relation (4.39), by Theorem 4.5. This, together with (4.29) and (4.11), imply that φ^j goes to 0 for $j = 0, \ldots, k$, in such way that $f_{k+1}(v^0, \ldots, v^k)$, given by (4.10), remains bounded in any closed domain $\bar{D}_{\kappa_k}(-1)$ with $\kappa_k < \sigma_k$, uniformly in t. Since $f_{k+1}(v^0, \ldots, v^k)$ is holomorphic in $D_{\sigma_k}(-1)$, by Theorem 4.3, Cauchy formula can be used to obtain upper bounds for its derivatives. By the definition (4.19), this establishes (4.28) for $i = k+1$ and completes the proof of this proposition.

□

4.2 Gevrey Estimates

The present subsection is devoted to the Gevrey estimates of $\{v^i(t,p)\}$, the coefficients of the formal power series (4.1). The majorant relation (4.39) does not specify how η_i depends on i and this dependency will be investigate using the system of equations (4.8).

By Theorems 4.3 and 4.5, $v^l(t,p)$ is holomorphic in $D_{\sigma_l}(-1)$ and $v^l(t,p) + \eta_l/\alpha(p)$ is bounded uniformly in $t \in \mathbb{R}_+$ for every $p \in \bar{D}_{\kappa_l}(-1)$ with

$$\cdots < \sigma_{l+1} \le \kappa_l < \sigma_l \le \cdots < \sigma_1 \le \kappa_0 < \sigma_0 < 1/2 \ .$$

To state our results, let $(\sigma_l)_{l\ge 0}$ and $(\kappa_l)_{l\ge 0}$ be sequences such that $\lim_{l\to\infty}\sigma_l = \sigma_\infty > 0$ exists (then, $\lim_{l\to\infty}\kappa_l$ also exists, let say $\kappa_\infty = \sigma_\infty$) satisfying

$$\sigma_1 - \sigma_\infty = \sum_{l=1}^{\infty}\left[(\sigma_l - \kappa_l) + (\kappa_l - \sigma_{l+1})\right] = 2\sum_{l=1}^{\infty}\frac{\delta}{l^\Delta}\ , \tag{4.41}$$

for some $\delta > 0$ and $\Delta > 1$. Note that $\sigma_1 - \sigma_\infty$, Δ and δ are not independent and they satisfy

$$\frac{\delta}{2}\frac{1}{\Delta - 1} \leq \sigma_1 - \sigma_\infty \leq \frac{\delta}{2}\frac{\Delta}{\Delta - 1}$$

So, δ goes to 0 if Δ approaches to 1.

Theorem 4.6 *Let $\left\{v^l(t,p)\right\}_{l\geq 0}$ be the solution of the system of equations (4.8) with the initial data $\left\{v_0^l(p)\right\}_{l\geq 0}$ for $\left\{\beta^l\right\}_{l\geq 0}$ at the critical value $\left\{\beta_c^l\right\}_{l\geq 0}$. Then, there exists a constant $0 < \varkappa < \infty$ so that*

$$\left|v^l(t,p)\right| \leq LB_l\varkappa^l \tag{4.42}$$

holds for every $p \in \bar{D}_{\kappa_l}(-1)$; the constant B_0 depends on σ_0 and, for $l \geq 1$,

$$B_l = \delta A \frac{1}{l^{2+\Delta}} l!^s$$

with $s \geq 1 + 2\Delta$ where Δ and A is as in (4.41) and Lemma 2.7; for every $0 < \sigma_0 < 1/2$, $L = L(t,\sigma_0) > 0$ is a linear function of t satisfying

$$\varrho := \max_{t\geq 0} \frac{L(t,\sigma_0)}{c(t,\sigma_0)} \leq \frac{1}{2} \tag{4.43}$$

where $c(t,\sigma_0) = \min_{p\in \bar{D}_{\sigma_0}(-1)} v^0(t,p)$ is positive and exhibit linear growth in t, by (4.29).

Remark 4.7 *Theorem 4.6 implies that the formal power series (4.1) is Gevrey of order $1 + 2\Delta$ where Δ can be chosen arbitrarily close to 1.*

Proof We assume that an estimate similar to (4.42) holds for Nagumo norm of order l of $v^l(t,p)$ (see (2.2), for definition):

$$\left\|v^l\right\|_l := \left\|v^l\right\|_{l,\sigma_l} \leq LB_l\mu^l \tag{4.44}$$

for $l = 0, 1, \ldots, i-1$, where μ is a constant related with $\varkappa$, to be especified later. We shall establish (4.44) for $l = i$, assuming, temporarily, that (4.43) holds. To begin with, we note that Cauchy's integral formula together with (4.42) and $p \in \bar{D}_{\kappa_l}(-1)$, yields

$$\begin{aligned}
\left|v_p^l(t,p)\right| &\leq \frac{1}{2\pi}\oint_{\mathcal{C}} \frac{\left|v^l(t,z)\right|}{\left(|1+z| - |1+p|\right)^2}\,|dz| \\
&\leq L\frac{B_l}{\sigma_l - \kappa_l}\mu_l^l
\end{aligned}$$

for some smooth Jordan curve $\mathcal{C}$ over $\bar{D}_{\sigma_l}(-1)$ containing p inside. Analogous estimate has been proven in Lemma 2.5 for Nagumo norm with varying domains:

$$\left\|v_p^l\right\|_{l,\kappa_{j+1}} \leq \frac{1}{\sigma_l - \kappa_{j+1}}\left\|v^l\right\|_l \leq \frac{l^\Delta}{\delta}\left\|v^l\right\|_l \tag{4.45}$$

holds for $l \leq j$. Note that the estimate keeps the order of Nagumo norm unchanged.

4.2.1 A recursive Relation

We set $C_l := l^\Delta B_l/\delta = Al!^s/l^2$ for $l \geq 1$ and note that such constants satisfy (2.7). Substituting (4.45) into (4.11), together with property 2. of Nagumo norms and (4.43), yields

$$\begin{aligned}
\left\|\varphi^j\right\|_{j,\kappa_{j+1}} &\leq \sum_{k=1}^{j} \left\|\frac{1}{v_p^0}\right\|_{0,\kappa_{j+1}}^{k+1} \sum_{\substack{l_1,\ldots,l_k\geq 1:\\ l_1+\cdots+l_k=j}} \left\|v_p^{l_1}\right\|_{l_1,\kappa_{j+1}} \cdots \left\|v_p^{l_k}\right\|_{l_k,\kappa_{j+1}} \\
&\leq \sum_{k=1}^{j} \left\|\frac{1}{v_p^0}\right\|_{0,\kappa_0}^{k+1} \sum_{\substack{l_1,\ldots,l_k\geq 1:\\ l_1+\cdots+l_k=j}} \frac{l_1^\Delta}{\delta} \left\|v^{l_1}\right\|_{l_1} \cdots \frac{l_k^\Delta}{\delta} \left\|v^{l_k}\right\|_{l_k} \\
&\leq \frac{\mu^j}{c} \sum_{k=1}^{j} \left(\frac{L}{c}\right)^k \sum_{\substack{l_1,\ldots,l_k\geq 1:\\ l_1+\cdots+l_k=j}} C_{l_1}\cdots C_{l_k} \\
&\leq \frac{1}{c} C_j \mu^j. \qquad (4.46)
\end{aligned}$$

for every $j = 1, 2, \ldots, i-1$. By Lemma 2.5 again, it follows analogously for each $j = 1, 2, \ldots, i-1$

$$\left\|\varphi_p^j\right\|_{j,\sigma_{j+1}} \leq \frac{1}{\sigma_{j+1}-\kappa_{j+1}} \left\|\varphi^j\right\|_{j,\kappa_{j+1}} \leq \frac{1}{c}\frac{j^\Delta C_j}{\delta}\mu^j \ , \qquad (4.47)$$

and (remembering that $\varphi^0 = -1/v_p^0$)

$$\left\|\varphi_p^0\right\|_{0,\sigma_{j+1}} \leq \frac{1}{\sigma_0 - \sigma_{j+1}} \left\|\varphi^0\right\|_{0,\sigma_0} \leq \frac{1}{c(\sigma_0-\kappa_1)} \, . \qquad (4.48)$$

Replacing these bounds in (4.10), yields

$$\begin{aligned}
\left\|f_i(v^0,\ldots,v^{i-1})\right\|_{i-1,\sigma_i} &\leq \left\|v^{i-1}\right\|_{i-1} \left\|\varphi_p^0\right\|_{0,\sigma_1} + \sum_{j=1}^{i-2} \left\|v^{i-1-j}\right\|_{i-1-j} \left\|\varphi_p^j\right\|_{j,\sigma_{j+1}} \\
&\qquad + \left\|v^0\right\|_{0,\sigma_0} \left\|\varphi_p^{i-1}\right\|_{i-1,\sigma_i} \\
&\leq \frac{1}{c}\left(\frac{1}{(i-1)^\Delta} + (i-2)^\Delta + \frac{(i-1)^\Delta B_0}{\delta}\right) C_{i-1}\nu_{i-1}^{i-1} \\
&\leq \frac{1}{c}\left(1+\frac{B_0}{\delta}\right)(i-1)^\Delta C_{i-1}\nu_{i-1}^{i-1} \qquad (4.49)
\end{aligned}$$

uniformly in $t \geq 0$, for every $i \geq 2$. In the first inequality we have used property 4. and (2.3) of Nagumo norms.

To estimate $v^i(t,p)$, β^i needs to be fixed at its critical value. We have seem that the exponentially increasing terms in the solution (4.31) of (4.26) are cancelled under the condition (4.34) and αv_0^i and αf_i have expanded in power series around $p = -1$ to perform the cancellation. The same cancellation will be exhibited using the first mean value theorem instead.

By (4.19) and (4.20), we have $\gamma_{-1}^i(t) = -2f_i\big|_{p=-1}$ and $c_{0,-1}^i = -2v_0^i(-1)$ and equation (4.34) can be conveniently written as

$$v_0^i(-1) + \int_0^\infty f_i(v^0(s,-1),\ldots,v^{i-1}(s,-1))e^{-2s}\,ds = 0\ . \tag{4.50}$$

Moreover, we write the ratio of $\alpha\circ\phi_t$ and α in both terms of the r.h.s. of (4.16) as

$$\frac{\alpha\circ\phi_t(p)}{\alpha(p)} = \frac{e^{2t}}{p^2} + R(t,p) \tag{4.51}$$

where, by (4.17) and (4.18),

$$\begin{aligned} R(t,p) &= e^{2t}\left(\frac{1}{(p-e^{-2t}(1+p))^2} - \frac{1}{p^2}\right) \\ &= \frac{1+p}{p^2}\frac{2p-e^{-2t}(1+p)}{(p-e^{-2t}(1+p))^2} \end{aligned}$$

is estimates by

$$\|R\|_{0,\sigma_i} \le \frac{\sigma_i + 2\sigma_i^2}{(1-\sigma_i)^4}\ , \tag{4.52}$$

uniformly in $t\ge 0$. We now apply the first mean value theorem (see e.g. [R]) to v_0^i and f_i:

$$\begin{aligned} v_0^i(p) &= v_0^i(-1) + A_0^i(1+p) \\ f_i &= f_i\big|_{p=-1} + B^i(1+p) \end{aligned} \tag{4.53}$$

where

$$\begin{aligned} A_0^i &= \left(v_0^i\right)'(-1+a(1+p)) \\ B^i &= \frac{df_i}{dp}\Big|_{p=-1+a'(1+p)} \end{aligned}$$

for some $0 < a, a' < 1$. Applying property 3. of Nagumo norms together with the fact that $-1 + a(1+p) \in \bar{D}_{\sigma_i}(-1)$ if $p\in\bar{D}_{\sigma_i}(-1)$, these quantities can be estimated as

$$\begin{aligned} \left\|A_0^i\right\|_i &\le ei\left\|v_0^i\right\|_{i-1,\sigma_i} \\ \left\|B^i\right\|_i &\le ei\left\|f_i\right\|_{i-1,\sigma_i}\ . \end{aligned}$$

In view of (4.50), (4.51) and 4.53), equation (4.16) can be written as

$$\begin{aligned} v^i(t,p) &= \frac{e^{2t}}{p^2}\left(A_0^i\ (1+\phi_t(p)) - \int_t^\infty f_i\big|_{p=-1}e^{-2s}\,ds + \int_0^t B^i\ (1+\phi_{t-s}(p))e^{-2s}\,ds\right) \\ &\quad +R(t,p)\left(v_0^i\circ\phi_t(p) + \int_0^t f_i\circ\phi_{t-s}(p)\,ds\right)\ . \end{aligned} \tag{4.54}$$

Since, by the first inequality of (4.27), $1+\phi_t(p)$ is an $O(e^{-2t})$ contractive map, it follows by property 2. of Nagumo norms and an estimation analogous to (4.35) that

$$\begin{aligned}
\left\| \frac{e^{2t}}{p^2} A_0^i \, (1+\phi_t(p)) \right\|_i &\leq \left\| \frac{e^{2t}}{p^2} (1+\phi_t(p)) \right\|_{0,\sigma_i} \left\| A_0^i \right\|_i \\
&\leq \frac{ei\sigma_i}{(1-\sigma_i)^3} \left\| v_0^i \right\|_{i-1,\sigma_i} \ , \qquad (4.55)
\end{aligned}$$

$$\begin{aligned}
\left\| \frac{e^{2t}}{p^2} \int_t^\infty f_i\big|_{p=-1} e^{-2s}\, ds \right\|_i &\leq \left\| \frac{e^{2t}}{p^2} \int_t^\infty e^{-2s}\, ds \right\|_{0,\sigma_i} \|f_i\|_i \\
&\leq \frac{1}{2(1-\sigma_i)^2} \|f_i\|_{i-1,\sigma_i} \qquad (4.56)
\end{aligned}$$

uniformly in $t \in \mathbb{R}_+$. Moreover, using

$$\int_0^t \frac{\sigma}{1-\sigma(1-e^{-2s})} ds = \frac{\sigma}{1-\sigma} \left(t + \frac{1}{2} \ln\left(1-\sigma(1-e^{-2t})\right) \right) \ ,$$

we have

$$\begin{aligned}
\left\| \frac{e^{2t}}{p^2} \int_0^t B_i \, (1+\phi_{t-s}(p)) e^{-2s}\, ds \right\|_i &\leq \left\| \frac{e^{2t}}{p^2} \int_0^t (1+\phi_{t-s}(p)) e^{-2s}\, ds \right\|_{0,\sigma_i} \|B_i\|_i \\
&\leq \frac{2t+\ln(1-\sigma_i)}{2(1-\sigma_i)^3} ei\sigma_i \|f_i\|_{i-1,\sigma_i} \ . \qquad (4.57)
\end{aligned}$$

Lastly, using (4.52), the second line of (4.54) can be estimated as

$$\begin{aligned}
\left\| R(t,p) \left(v_0^i \circ \phi_t(p) + \int_0^t f_i \circ \phi_{t-s}(p)\, ds \right) \right\|_i &\leq \|R(t,p)\|_{0,\sigma_i} \left(\left\| v_0^i \circ \phi_t \right\|_i + \int_0^t \|f_i \circ \phi_{t-s}\|_i \, ds \right) \\
&\leq \frac{\sigma_i + 2\sigma_i^2}{(1-\sigma_i)^4} \left(\left\| v_0^i \right\|_{i-1,\sigma_i} + t \|f_i\|_{i-1,\sigma_i} \right) \qquad (4.58)
\end{aligned}$$

where, by definition (2.2) together with contractivity (4.27),

$$\begin{aligned}
\left\| v_0^i \circ \phi_t \right\|_i &= \sup_{p \in D_{\sigma_i}(-1)} d_i(p) \left| v_0^i \circ \phi_t(p) \right| \\
&\leq \sup_{p \in D_{\sigma_i}(-1)} \frac{d_i(p)}{d_i(\phi_t(p))} \left\| v_0^i \right\|_i \\
&\leq \left\| v_0^i \right\|_{i-1,\sigma_i}
\end{aligned}$$

(analogously for $\|f_i \circ \phi_{t-s}\|_i$) in view of

$$\begin{aligned}
\sup_{p\in D_{\sigma_i}(-1)} \frac{d_i(p)}{d_i(\phi_t(p))} &= \sup_{p\in D_{\sigma_i}(-1)} \frac{\sigma_i - |1+p|}{\sigma_i - \left|\dfrac{e^{-2t}\,(1+p)}{1-(1+p)\,(1-e^{-2t})}\right|} \\
&\leq \sup_{p\in D_{\sigma_i}(-1)} \frac{\sigma_i - |1+p|}{\sigma_i - \dfrac{e^{-2t}\,|1+p|}{1-|1+p|\,(1-e^{-2t})}} \leq 1 \ ,
\end{aligned}$$

uniformly in $t \in \mathbb{R}_+$.

Replacing (4.55)–(4.58) into the nagumo norm of (4.54), we conclude (at criticality)

$$\left\|v^i\right\|_i \leq D_i \|v_0^i\|_{i-1,\sigma_i} + E_i \|f_i\|_{i-1,\sigma_i} \tag{4.59}$$

where

$$\begin{aligned}
D_i &= \frac{ei\sigma_i}{(1-\sigma_i)^3} + \frac{\sigma_i + 2\sigma_i^2}{(1-\sigma_i)^4} \\
E_i &= \frac{1}{2(1-\sigma_i)^2} + \frac{ei\sigma_i \ln(1-\sigma_i)}{2(1-\sigma_i)^3} + \left(\frac{ei\sigma_i}{(1-\sigma_i)^3} + \frac{\sigma_i + 2\sigma_i^2}{(1-\sigma_i)^4}\right) t
\end{aligned}$$

and the nagumo norms $\|v_0^i\|_{i-1,\sigma_i}$ and $\|f_i\|_{i-1,\sigma_i}$ satisfy, respectively,[8]

$$\|v_0^i\|_{i-1,\sigma_i} \leq L_0 \frac{Ai!^{s'}}{i^2} \nu^i$$

(see (3.22) and Remark 4.2) and (4.49). Hence

$$\left\|v^i\right\|_i \leq LB_i \nu_i^i$$

holds provided

$$\frac{1}{L\delta}\left[D_i L_0 i^{\Delta - s'} \left(\frac{\nu}{\mu}\right)^i i!^{s'-s} + E_i \left(1 + \frac{B_0}{2\delta}\right) \frac{i^{2+\Delta}}{(i-1)^{2-\Delta} i^s} \frac{1}{c\mu}\right] \leq 1 \tag{4.60}$$

is satisfied for all $t \geq 0$ and $i \geq 2$. Condition (4.60) is true if we pick $\mu > \nu$ large enough and $s \geq 1 + \max(s', 2\Delta)$ (recall that D_i and E_i increases with i).

[8] Since $v_0 = (u_0')^{-1} = y_0^{-1}$, by Theorem 3.3 of [CM1] all constants in the Gevrey estimate (3.22), except the holomorphic domain, are preserved by the inverse function theorem. The holomorphic domain of v_0 has been directly estimate from equation (4.6), instead.

Remark 4.8 *A recursive relation can be set for* $\|v^l\|_l$. *From equation (4.59), first line of (4.49), equation (4.47) and second line of (4.46), we have*

$$\begin{aligned} \|v_0^i\|_i &\le \eta_{0,i} \\ \|v^i\|_i &\le \eta_i \end{aligned}$$

with

$$\eta_i = D_i \eta_{0,i} + E_i \sum_{j=0}^{i-1} \frac{(j+1)^\Delta}{\delta} \sum_{k=1}^{j} \frac{1}{c^{k+1}} \sum_{\substack{l_1,\dots,l_k \ge 1: \\ l_1+\cdots+l_k=j}} \frac{l_1^\Delta}{\delta}\eta_{l_1} \cdots \frac{l_k^\Delta}{\delta}\eta_{l_k} \ .$$

4.2.2 Setting of Parameters

Verifying (4.44) for $l = 0$ By Theorem 4.1, $v^0(t,p)$ is a holomorphic function in $\mathbb{R}_+ \times D_1(-1)$. If we consider any $\sigma_0 < 1/2$, the *Maximum Modulus Principle* (see e.g. [H]) tells us that the maximum of $|v^0(t,p)|$, for $p \in \bar{D}_{\sigma_0}(-1)$, is attained at some p at the border of $\bar{D}_{\sigma_0}(-1)$. In the other hand, fixing $\beta_0 = 4$ in (4.12) and expanding it around $p = -1$ (see (4.38), for notation), we see that $a_0^0(t) = 1/2$, $a_1^0(t) = 3/2 + 2t$ and

$$a_n^0(t) = \frac{2n+1}{2} + (n^2+n)t - \frac{1}{2}\sum_{k=1}^{n-1} \frac{k(k+1)}{n-k}(1-e^{-2t})^{n-k}$$

for $n \ge 2$. So, $a_n^{(0)}(t) > 0$ for all $n \ge 0$ and $t \ge 0$.

From these two facts, it follows that

$$\begin{aligned} \|v^0\|_0 &= v^0(t,p)\big|_{1+p=\sigma_0} \\ &= \frac{1+\sigma_0}{2(1-\sigma_0)^2} + \frac{\sigma_0}{(1-\sigma_0)^3}\left[2t + \ln\left(1-\sigma_0(1-e^{-2t})\right)\right] \\ &\le \frac{1+\sigma_0}{2(1-\sigma_0)^2} + \frac{\sigma_0}{(1-\sigma_0)^3}\, 2t := B_0 L \ . \end{aligned} \tag{4.61}$$

Estimating $c(t,\sigma_0)$ By the argument after (4.12), $v_p^0(t,p) \neq 0$ in $\mathbb{R}_+ \times \bar{D}_{\sigma_0}(-1)$. Hence, the *Minimum Modulus Principle[H]* tells us that the minimum of $|v_p^0(t,p)|$, for $p \in \bar{D}_{\sigma_0}(-1)$, is attained at some p at the border of $\bar{D}_{\sigma_0}(-1)$. Since

$$v_p^0(t,p) = \sum_{n=0}^{\infty}(n+1)a_{n+1}^0(t)\,(1+p)^n$$

and $a_n^{(0)}(t) > 0$ for all $n \ge 0$ and $t \ge 0$, we have

$$\min_{p \in \bar{D}_{\sigma_0}(-1)} \left|v_p^0(t,p)\right| = v_p^0(t,p)\big|_{1+p=-\sigma_0}$$

and

$$
\begin{aligned}
c(t,\sigma_0) &= \frac{3-\sigma_0}{2(1+\sigma_0)^3} + \frac{1-2\sigma_0}{(1+\sigma_0)^4}\left\{2t + \ln\left[1+\sigma_0(1-e^{-2t})\right]\right\} \\
&\quad + \frac{\sigma_0(1-e^{-2t})}{(1+\sigma_0)^3[1+\sigma_0(1-e^{-2t})]} \\
&\geq \frac{3-\sigma_0}{2(1+\sigma_0)^3} + \frac{1-2\sigma_0}{(1+\sigma_0)^4}\, 2t\ . \qquad (4.62)
\end{aligned}
$$

Verifying (4.44) for $l = 1$ By Theorem 4.1, $f_1(v^0)$ (see expression 3.14) is a holomorphic function in $\mathbb{R}_+ \times \bar{D}_{\sigma_0}(-1)$ which, together with (4.48) and property 4. of Nagumo norms, can be estimate by

$$
\begin{aligned}
\|f_1\|_1 &= \sigma_1 \left\|-1 + v^0\varphi_p^0\right\|_{0,\sigma_1} \\
&\leq \sigma_1 + \frac{\sigma_1 B_0}{c(\sigma_0-\sigma_1)}\ .
\end{aligned}
$$

From this bound, together with (4.59), we conclude that (at criticality)

$$
\left\|v^1\right\|_1 \leq D_1 \left\|v_0^1\right\|_1 + E_1 \left\|f_1\right\|_1 \leq L B_1 \mu
$$

holds provided that

$$
\mu \geq \max_{t\geq 0} \frac{1}{L\delta}\left[D_1 L_0 \nu + \frac{E_1}{A}\left(1 + \frac{\sigma_1 B_0}{\sigma_0 - \sigma_1}\right)\right]\ . \qquad (4.63)
$$

The parameter σ_1 is free but we shall fix it after choosing σ_0. μ is taken so large that (4.60), for all $i \geq 2$, and 4.63) hold.

Verifying (4.43) Once we have an expression for $B_0 L$ and have estimated c, it remains for us to ajust B_0 in such way that (4.43) is satisfied. By (4.61) and (4.62), we have

$$
\rho = \max_{t\geq 0} \frac{L(t,\sigma_0)}{c(t,\sigma_0)} \leq \frac{1}{B_0} \max_{t\geq 0} \chi(t,\sigma_0)
$$

where

$$
\chi = \frac{\dfrac{1+\sigma_0}{2(1-\sigma_0)^2} + \dfrac{2\sigma_0}{(1-\sigma_0)^3}t}{\dfrac{3-\sigma_0}{2(1+\sigma_0)^3} + \dfrac{2(1-2\sigma_0)}{(1+\sigma_0)^4}t} := \frac{\alpha + \beta t}{\gamma + \delta t}
$$

attains its maximum value either at $t^* = 0$ (if $\alpha/\gamma > \beta/\delta$) or at $t^* = \infty$ (if $\alpha/\gamma < \beta/\delta$). Hence,

$$\max_{t \geq 0} \chi(t, \sigma_0) = \max \left(\frac{\alpha}{\gamma}, \frac{\beta}{\delta} \right) \equiv \chi^*(\sigma_0)$$

and $B_0 = B_0(\sigma_0)$ is chosen so that $\rho \leq 1/2$ holds for every $0 < \sigma_0 < 1/2$:

$$B_0 := 2\chi^*(\sigma_0)$$

Note that β/δ crossover α/γ at $\sigma_0 = 0.407252$ and $\chi^*(\sigma_0)$ is the convex envelop of the two graphics, shown in the figure below. In this manner, (4.43) is guaranteed and we set

$$L(t, \sigma_0) := \frac{1}{2\chi^*(\sigma_0)} \left(\frac{1 + \sigma_0}{2(1 - \sigma_0)^2} + \frac{\sigma_0}{(1 - \sigma_0)^3} 2t \right)$$

for every $t \in \mathbb{R}_+$.

4.2.3 Concluding the Proof

We have established, by induction, the Gevrey estimate:

$$\sup_{p \in D_{\sigma_l}(-1)} (d_{\sigma_l}(p))^l \left| v^l(t, p) \right| \equiv \|v^l\|_l \leq L B_l \mu^l \ , \qquad \forall\, l \geq 0 \ .$$

Now, we complete the proof applying the same procedure used in the proof of Lemma 3.3. By property 5 of Nagumo norms,

$$\left| v^l(t, p) \right| \leq \frac{1}{(\sigma_l - \varsigma)^l} \|v^l\|_l \leq L B_l \varkappa^l$$

holds for all $l \geq 0$ and $t \in \mathbb{R}_+$, uniformly in $\bar{D}_\varsigma(0)$ for some $\varsigma < \sigma_\infty$, with $\varkappa = \mu/(\sigma_\infty - \varsigma)$, which concludes the proof of Theorem 4.6.

$\square$

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