3 The Ising Model

In this chapter, we study the Ising model on $\mathbb{Z}^d$, which was introduced informally in Section 1.4.2. We provide both precise definitions of the concepts involved and a detailed analysis of the conditions ensuring the existence or absence of a phase transition in this model, therefore providing full rigorous justification to the discussion in Section 1.4.3. Namely,

- In Section 3.1, the Ising model on $\mathbb{Z}^d$ is defined, together with various types of boundary conditions.
- In Section 3.2, several concepts of fundamental importance are introduced, including: the thermodynamic limit, the pressure and the magnetization. The latter two quantities are then computed explicitly in the case of the one-dimensional model (Section 3.3).
- The notion of infinite-volume Gibbs state is given a precise meaning in Section 3.4. In Section 3.6, we discuss correlation inequalities, which play a central role in the analysis of ferromagnetic systems like the Ising model.
- In Section 3.7, the phase diagram of the model is analyzed in detail. In particular, several criteria for the presence of first-order phase transitions, based on the magnetization and the pressure of the model, are introduced in Section 3.7.1. The latter are used to prove the existence of a phase transition when $h = 0$ (Sections 3.7.2 and 3.7.3) and the absence of a phase transition when $h \neq 0$ (Section 3.7.4). A summary with a link to the discussion in the Introduction is given in Section 3.7.5.
- Finally, in Section 3.10, the reader can find a series of complements to this chapter, in which a number of interesting topics, related to the core of the chapter but usually more advanced or specific, are discussed in a somewhat less precise manner.

We emphasize that some of the ideas and concepts introduced in this chapter are not only useful for the Ising model, but are also of central importance for statistical mechanics in general. They are thus fundamental for the understanding of other parts of the book.

Revised version, August 22 2017
To be published by Cambridge University Press (2017)

3.1 Finite-volume Gibbs distributions

In this section, the Ising model on $\mathbb{Z}^d$ is defined precisely and some of its basic properties are established. As a careful reader might notice, some of the definitions in this chapter differ slightly from those of Chapter 1. This is done for later convenience.

▶ **Finite volumes with free boundary condition.** The configurations of the Ising model in a finite volume $\Lambda \Subset \mathbb{Z}^d$ with free boundary condition are the elements of the set

$$\Omega_\Lambda \stackrel{\text{def}}{=} \{-1,1\}^\Lambda .$$

A configuration $\omega \in \Omega_\Lambda$ is thus of the form $\omega = (\omega_i)_{i\in\Lambda}$. The basic random variable associated to the model is the **spin** at a vertex $i \in \mathbb{Z}^d$, which is the random variable $\sigma_i : \Omega_\Lambda \to \{-1,1\}$ defined by $\sigma_i(\omega) \stackrel{\text{def}}{=} \omega_i$.

We will often identify a finite set Λ with the graph that contains all edges formed by nearest-neighbor pairs of vertices of Λ. We denote the latter set of edges by

$$\mathscr{E}_\Lambda \stackrel{\text{def}}{=} \bigl\{\{i,j\} \subset \Lambda : i \sim j\bigr\} .$$

To each configuration $\omega \in \Omega_\Lambda$, we associate its **energy**, given by the Hamiltonian

$$\mathscr{H}^{\varnothing}_{\Lambda;\beta,h}(\omega) \stackrel{\text{def}}{=} -\beta \sum_{\{i,j\}\in\mathscr{E}_\Lambda} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i\in\Lambda} \sigma_i(\omega) ,$$

where $\beta \in \mathbb{R}_{\geq 0}$ is the inverse temperature and $h \in \mathbb{R}$ is the magnetic field. The superscript $\varnothing$ indicates that this model has **free boundary condition**: spins in Λ do not interact with other spins located outside of Λ.

Definition 3.1. *The Gibbs distribution of the Ising model in Λ with* ***free boundary condition****, at parameters β and h, is the distribution on Ω_Λ defined by*

$$\mu^{\varnothing}_{\Lambda;\beta,h}(\omega) \stackrel{\text{def}}{=} \frac{1}{\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}} \exp\bigl(-\mathscr{H}^{\varnothing}_{\Lambda;\beta,h}(\omega)\bigr) .$$

The normalization constant

$$\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h} \stackrel{\text{def}}{=} \sum_{\omega\in\Omega_\Lambda} \exp\bigl(-\mathscr{H}^{\varnothing}_{\Lambda;\beta,h}(\omega)\bigr)$$

is called the ***partition function*** *in Λ with free boundary condition.*

▶ **Finite volumes with periodic boundary condition.** We now consider the Ising model on the torus $\mathbb{T}_n$, defined as follows. Its set of vertices is given by

$$V_n \stackrel{\text{def}}{=} \{0,\dots,n-1\}^d ,$$

and there is an edge between each pair of vertices $i = (i_1,\dots,i_d)$, $j = (j_1,\dots,j_d)$ such that $\sum_{r=1}^d |(i_r - j_r) \mod n| = 1$; see Figure 3.1 for illustrations in dimensions 1 and 2. We denote by $\mathscr{E}^{\text{per}}_{V_n}$ the set of edges of $\mathbb{T}_n$.

Configurations of the model are now the elements of $\{-1,1\}^{V_n}$ and have an energy given by

$$\mathscr{H}^{\text{per}}_{V_n;\beta,h}(\omega) \stackrel{\text{def}}{=} -\beta \sum_{\{i,j\}\in\mathscr{E}^{\text{per}}_{V_n}} \sigma_i(\omega)\sigma_j(\omega) - h \sum_{i\in V_n} \sigma_i(\omega) .$$

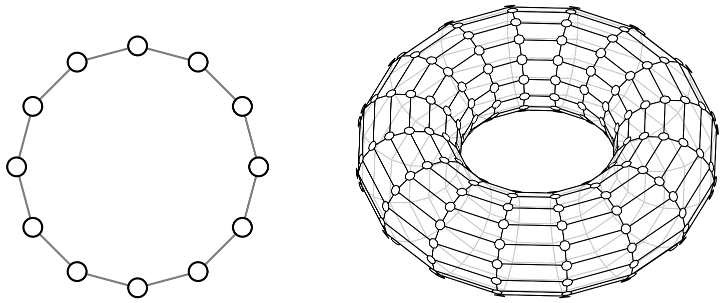


Figure 3.1: Left: the one-dimensional torus $\mathbb{T}_{12}$. Right: the two-dimensional torus $\mathbb{T}_{16}$.

Definition 3.2. *The Gibbs distribution of the Ising model in V_n with **periodic boundary condition**, at parameters β and h, is the probability distribution on $\{-1,1\}^{V_n}$ defined by*

$$\mu^{\mathrm{per}}_{V_n;\beta,h}(\omega) \stackrel{\mathrm{def}}{=} \frac{1}{\mathbf{Z}^{\mathrm{per}}_{V_n;\beta,h}} \exp\bigl(-\mathscr{H}^{\mathrm{per}}_{V_n;\beta,h}(\omega)\bigr).$$

The normalization constant

$$\mathbf{Z}^{\mathrm{per}}_{V_n;\beta,h} \stackrel{\mathrm{def}}{=} \sum_{\omega\in\Omega_{V_n}} \exp\Bigl(-\mathscr{H}^{\mathrm{per}}_{V_n;\beta,h}(\omega)\Bigr)$$

*is called the **partition function** in V_n with periodic boundary condition.*

► **Finite volumes with configurations as boundary condition.** It will turn out to be useful to consider the Ising model on the full lattice $\mathbb{Z}^d$, but with configurations which are frozen outside a finite set.

Let us thus consider configurations of the Ising model on the infinite lattice $\mathbb{Z}^d$, that is, elements of

$$\Omega \stackrel{\mathrm{def}}{=} \{-1,1\}^{\mathbb{Z}^d}.$$

Fixing a finite set $\Lambda \Subset \mathbb{Z}^d$ and a configuration $\eta \in \Omega$, we define a **configuration of the Ising model in Λ with boundary condition η** as an element of the finite set

$$\Omega^\eta_\Lambda \stackrel{\mathrm{def}}{=} \bigl\{\omega\in\Omega : \omega_i = \eta_i\,, \forall i \notin \Lambda\bigr\}.$$

The **energy** of a configuration $\omega \in \Omega^\eta_\Lambda$ is defined by

$$\mathscr{H}_{\Lambda;\beta,h}(\omega) \stackrel{\mathrm{def}}{=} -\beta \sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda} \sigma_i(\omega)\sigma_j(\omega) - h\sum_{i\in\Lambda}\sigma_i(\omega)\,, \tag{3.1}$$

where we have introduced

$$\mathscr{E}^{\mathrm{b}}_\Lambda \stackrel{\mathrm{def}}{=} \bigl\{\{i,j\}\subset\mathbb{Z}^d : \{i,j\}\cap\Lambda\neq\varnothing,\, i\sim j\bigr\}. \tag{3.2}$$

Note that $\mathscr{E}^{\mathrm{b}}_\Lambda$ differs from $\mathscr{E}_\Lambda$ by the addition of all the edges connecting vertices inside Λ to their neighbors outside Λ.

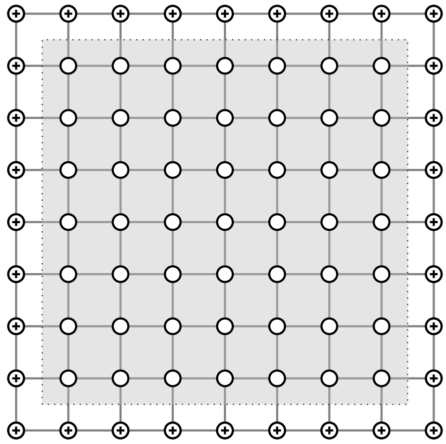


Figure 3.2: The model in a box Λ (shaded) with + boundary condition.

Definition 3.3. *The Gibbs distribution of the Ising model in* Λ ***with boundary condition*** $\boldsymbol{\eta}$*, at parameters* β *and* h*, is the probability distribution on* Ω_Λ^η *defined by*

$$\mu^\eta_{\Lambda;\beta,h}(\omega) \stackrel{\text{def}}{=} \frac{1}{\mathbf{Z}^\eta_{\Lambda;\beta,h}} \exp\bigl(-\mathscr{H}_{\Lambda;\beta,h}(\omega)\bigr)\,.$$

The normalization constant

$$\mathbf{Z}^\eta_{\Lambda;\beta,h} \stackrel{\text{def}}{=} \sum_{\omega\in\Omega^\eta_\Lambda} \exp\bigl(-\mathscr{H}_{\Lambda;\beta,h}(\omega)\bigr)$$

is called the ***partition function*** *with* η*-boundary condition.*

It will be seen later (in particular in Chapter 6) why defining $\mu^\eta_{\Lambda;\beta,h}$ on configurations in infinite volume is convenient (here, we could as well have defined it on Ω_Λ and included the effect of the boundary condition in the Hamiltonian).

Two boundary conditions play a particularly important role in the analysis of the Ising model: the + **boundary condition** η^+, for which $\eta^+_i \stackrel{\text{def}}{=} +1$ for all i (see Figure 3.2), and the − **boundary condition** η^-, similarly defined by $\eta^-_i \stackrel{\text{def}}{=} -1$ for all i. The corresponding Gibbs distributions will be simply denoted by $\mu^+_{\Lambda;\beta,h}$ and $\mu^-_{\Lambda;\beta,h}$; similarly, we will write $\Omega^+_\Lambda, \Omega^-_\Lambda$ for the corresponding sets of configurations.

On the notations used below. In the following, we will use the symbol # to denote a generic type of boundary condition. For instance, $\mathbf{Z}^\#_{\Lambda;\beta,h}$ can denote $\mathbf{Z}^\varnothing_{\Lambda;\beta,h}$, $\mathbf{Z}^{\text{per}}_{\Lambda;\beta,h}$ or $\mathbf{Z}^\eta_{\Lambda;\beta,h}$. In the case of periodic boundary condition, Λ will always implicitly be assumed to be a cube (see below).

Following the custom in statistical physics, expectation of a function f with respect to a probability distribution μ will be denoted by a bracket: $\langle f\rangle_\mu$. When the distribution is identified by indices, we will apply the same indices to the bracket.

For example, expectation of a function f under $\mu^{\#}_{\Lambda;\beta,h}$ will be denoted by

$$\langle f\rangle^{\#}_{\Lambda;\beta,h} \stackrel{\text{def}}{=} \sum_{\omega\in\Omega^{\#}_{\Lambda}} f(\omega)\mu^{\#}_{\Lambda;\beta,h}(\omega)\,.$$

We will often use $\langle\cdot\rangle^{\#}_{\Lambda;\beta,h}$ and $\mu^{\#}_{\Lambda;\beta,h}(\cdot)$ interchangeably.

3.2 Thermodynamic limit, pressure and magnetization

3.2.1 Convergence of subsets

It is well known that various statements in probability theory, such as the strong law of large numbers or the ergodic theorem, take on a much cleaner form when considering infinite samples. For the same reason, it is convenient to have some notion of Gibbs distribution for the Ising model on the whole of $\mathbb{Z}^d$. The theory describing *Gibbs measures* of infinite lattice systems will be discussed in detail in Chapter 6.

In this chapter, we adopt a more elementary point of view, using a procedure which consists in approaching an infinite system by a sequence of growing sets. This procedure, crucial for a proper description of thermodynamics and phase transitions, is called the *thermodynamic limit*.

To define the Ising model on the whole lattice $\mathbb{Z}^d$ (one often says "in infinite volume"), the thermodynamic limit will be considered along sequences of finite subsets $\Lambda_n \Subset \mathbb{Z}^d$ which **converge to $\mathbb{Z}^d$**, denoted by $\Lambda_n \uparrow \mathbb{Z}^d$, in the sense that

1. Λ_n is *increasing*: $\Lambda_n \subset \Lambda_{n+1}$,
2. Λ_n *invades* $\mathbb{Z}^d$: $\bigcup_{n\geq 1}\Lambda_n = \mathbb{Z}^d$.

Sometimes, in order to control the influence of the boundary condition and of the shape of the box on thermodynamic quantities, it will be necessary to impose a further regularity property on the sequence Λ_n. We will say that a sequence $\Lambda_n \uparrow \mathbb{Z}^d$ **converges to $\mathbb{Z}^d$ in the sense of van Hove**, which we denote by $\Lambda_n \Uparrow \mathbb{Z}^d$, if and only if

$$\lim_{n\to\infty} \frac{|\partial^{\text{in}}\Lambda_n|}{|\Lambda_n|} = 0\,, \tag{3.3}$$

where $\partial^{\text{in}}\Lambda \stackrel{\text{def}}{=} \{i\in\Lambda : \exists j\not\in\Lambda, j\sim i\}$. The simplest sequence to satisfy this condition is the sequence

$$\mathsf{B}(n) \stackrel{\text{def}}{=} \{-n,\dots,n\}^d\,.$$

Exercise 3.1. *Show that $\mathsf{B}(n) \Uparrow \mathbb{Z}^d$. Give an example of a sequence Λ_n that converges to $\mathbb{Z}^d$, but not in the sense of van Hove.*

3.2.2 Pressure

The partition functions introduced above play a very important role in the theory, in particular because they give rise to the pressure of the model.

Definition 3.4. *The **pressure in** $\Lambda \Subset \mathbb{Z}^d$, with boundary condition of the type #, is defined by*

$$\psi^{\#}_{\Lambda}(\beta,h) \stackrel{\text{def}}{=} \frac{1}{|\Lambda|}\log \mathbf{Z}^{\#}_{\Lambda;\beta,h}\,.$$

Exercise 3.2. *Show that, for all* $\Lambda \Subset \mathbb{Z}^d$*, all* $\beta \geq 0$ *and all* $h \in \mathbb{R}$*,*

$$\psi_\Lambda^{\varnothing}(\beta, h) = \psi_\Lambda^{\varnothing}(\beta, -h)\,, \quad \psi_\Lambda^{\mathrm{per}}(\beta, h) = \psi_\Lambda^{\mathrm{per}}(\beta, -h)\,, \quad \psi_\Lambda^{+}(\beta, h) = \psi_\Lambda^{-}(\beta, -h)\,.$$

The following simple observation will play an important role in the sequel.

Lemma 3.5. *For each type of boundary condition #,* $(\beta, h) \mapsto \psi_\Lambda^{\#}(\beta, h)$ *is convex.*

Proof. We consider $\psi_\Lambda^{\eta}(\beta, h)$, but the other cases are similar. Let $\alpha \in [0,1]$. Since $\mathscr{H}_{\Lambda;\beta,h}$ is an affine function of the pair (β, h), Hölder's inequality (see Appendix B.1.1) yields

$$\begin{aligned} \mathbf{Z}^{\eta}_{\Lambda;\alpha\beta_1+(1-\alpha)\beta_2,\alpha h_1+(1-\alpha)h_2} &= \sum_{\omega\in\Omega_\Lambda^\eta} e^{-\alpha\mathscr{H}_{\Lambda;\beta_1,h_1}(\omega)-(1-\alpha)\mathscr{H}_{\Lambda;\beta_2,h_2}(\omega)} \\ &\leq \Big(\sum_{\omega\in\Omega_\Lambda^\eta} e^{-\mathscr{H}_{\Lambda;\beta_1,h_1}(\omega)}\Big)^{\alpha}\Big(\sum_{\omega\in\Omega_\Lambda^\eta} e^{-\mathscr{H}_{\Lambda;\beta_2,h_2}(\omega)}\Big)^{(1-\alpha)}. \end{aligned}$$

Therefore, ψ_Λ^η is convex:

$$\psi_\Lambda^\eta\big(\alpha\beta_1 + (1-\alpha)\beta_2, \alpha h_1 + (1-\alpha)h_2\big) \leq \alpha\psi_\Lambda^\eta(\beta_1, h_1) + (1-\alpha)\psi_\Lambda^\eta(\beta_2, h_2)\,. \qquad \square$$

Of course, the finite-volume pressure $\psi_\Lambda^{\#}$ depends on Λ and on the boundary condition used. However, as the following theorem shows, when Λ is so large that $|\Lambda| \gg |\partial\Lambda|$, the boundary condition and the shape of Λ only provide negligible corrections: there exists a function $\psi(\beta, h)$ such that

$$\psi_\Lambda^{\#}(\beta, h) = \psi(\beta, h) + O(|\partial\Lambda|/|\Lambda|)\,.$$

$\psi(\beta, h)$ then provides a better candidate for the corresponding thermodynamic potential, since the latter does not depend on the "details" of the observed system, such as its shape.

Theorem 3.6. *In the thermodynamic limit, the* ***pressure***

$$\psi(\beta, h) \stackrel{\mathrm{def}}{=} \lim_{\Lambda\Uparrow\mathbb{Z}^d} \psi_\Lambda^{\#}(\beta, h)$$

is well defined and independent of the sequence $\Lambda \Uparrow \mathbb{Z}^d$ *and of the type of boundary condition. Moreover,* ψ *is convex (as a function on* $\mathbb{R}_{\geq 0} \times \mathbb{R}$*) and is even as a function of* h*.*

Proof. ► *Existence of the limit.* We start by proving convergence in the case of free boundary condition. The proof is done in two steps. We will first show existence of the limit

$$\lim_{n\to\infty} \psi_{D_n}^{\varnothing}(\beta, h)\,,$$

where $D_n \stackrel{\mathrm{def}}{=} \{1, 2, \ldots, 2^n\}^d$. After that, we extend the convergence to any sequence $\Lambda_n \Uparrow \mathbb{Z}^d$. Since the pair (β, h) is fixed, we will omit it from the notations most of the time, until the end of the proof.

The pressure associated to the box D_{n+1} will be shown to be close to the one associated to the box D_n. Indeed, let us decompose D_{n+1} into 2^d disjoint translates of D_n, denoted by $D_n^{(1)}, \ldots, D_n^{(2^d)}$:

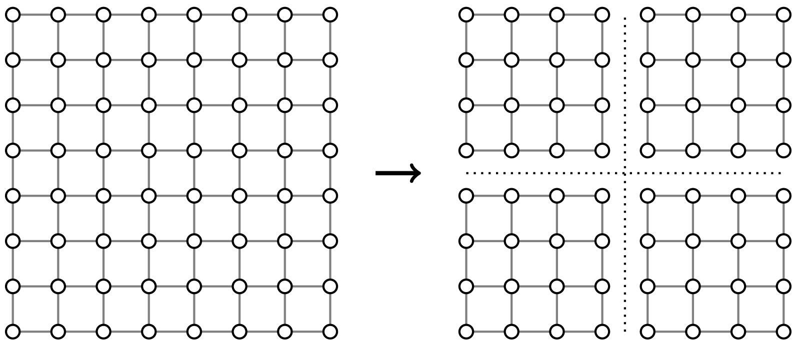


Figure 3.3: A cube D_{n+1} and its partition into 2^d translates of D_n. The interaction between different sub-boxes is denoted by $R_n(\omega)$.

The energy of ω in D_{n+1} can be written as

$$\mathscr{H}^{\varnothing}_{D_{n+1}} = \sum_{i=1}^{2^d} \mathscr{H}^{\varnothing}_{D_n^{(i)}} + R_n ,$$

where R_n represents the energy of interaction between pairs of spins that belong to different sub-boxes. Since each face of D_{n+1} contains $(2^{n+1})^{d-1}$ points, we have $|R_n(\omega)| \leq \beta d\,(2^{n+1})^{d-1}$. To obtain an upper bound on the partition function, we can write $\mathscr{H}^{\varnothing}_{D_{n+1}} \geq -\beta d\,(2^{n+1})^{d-1} + \sum_{i=1}^{2^d} \mathscr{H}^{\varnothing}_{D_n^{(i)}}$, which yields

$$\mathbf{Z}^{\varnothing}_{D_{n+1}} \leq e^{\beta d 2^{(n+1)(d-1)}} \sum_{\omega\in\Omega_{D_{n+1}}} \prod_{i=1}^{2^d} \exp\bigl(-\mathscr{H}^{\varnothing}_{D_n^{(i)}}(\omega)\bigr).$$

Splitting the sum over $\omega \in D_{n+1}$ into 2^d sums over $\omega^{(i)} \in D_n^{(i)}$,

$$\sum_{\omega\in\Omega_{D_{n+1}}} \prod_{i=1}^{2^d} \exp\bigl(-\mathscr{H}^{\varnothing}_{D_n^{(i)}}(\omega)\bigr) = \prod_{i=1}^{2^d} \sum_{\omega^{(i)}\in\Omega_{D_n^{(i)}}} \exp\bigl(-\mathscr{H}^{\varnothing}_{D_n^{(i)}}(\omega^{(i)})\bigr) = \Bigl(\mathbf{Z}^{\varnothing}_{D_n}\Bigr)^{2^d} ,$$

where we have used the fact that $\mathbf{Z}^{\varnothing}_{D_n^{(i)}} = \mathbf{Z}^{\varnothing}_{D_n}$ for all i. A lower bound can be obtained in a similar fashion, leading to

$$e^{-\beta d 2^{(n+1)(d-1)}} \Bigl(\mathbf{Z}^{\varnothing}_{D_n}\Bigr)^{2^d} \leq \mathbf{Z}^{\varnothing}_{D_{n+1}} \leq e^{\beta d 2^{(n+1)(d-1)}} \Bigl(\mathbf{Z}^{\varnothing}_{D_n}\Bigr)^{2^d} .$$

After taking the logarithm, dividing by $|D_{n+1}| = 2^{d(n+1)}$ and taking n large enough,

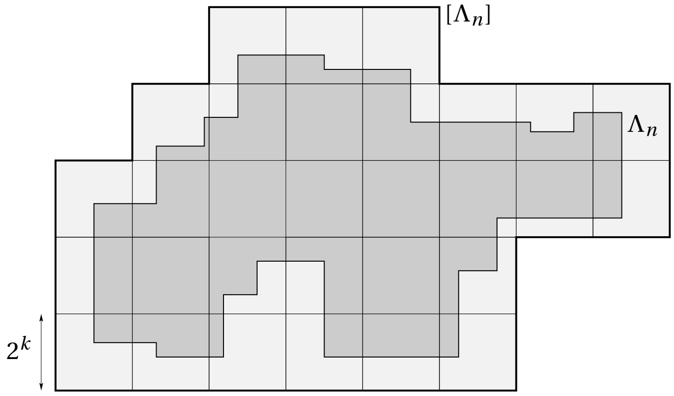
$$|\psi^{\varnothing}_{D_{n+1}} - \psi^{\varnothing}_{D_n}| \leq \beta\, d\, 2^{-(n+1)} .$$

This implies that ψ_{D_n} is a Cauchy sequence: for all $n \leq m$,

$$|\psi^{\varnothing}_{D_m} - \psi^{\varnothing}_{D_n}| \leq \beta\, d \sum_{k=n+1}^{m} 2^{-k} = \beta d (2^{-n} - 2^{-m}) .$$

Therefore, $\lim_{n\to\infty} \psi^{\varnothing}_{D_n}$ exists; we denote it by ψ.

Let us now consider an arbitrary sequence $\Lambda_n \Uparrow \mathbb{Z}^d$. We fix some integer k and consider a partition of $\mathbb{Z}^d$ into adjacent disjoint translates of D_k. For each n, consider a minimal covering of Λ_n by elements $D_k^{(j)}$ of the partition, and let $[\Lambda_n] \stackrel{\text{def}}{=} \bigcup_j D_k^{(j)}$:



We use the estimate

$$|\psi^{\varnothing}_{\Lambda_n} - \psi| \le |\psi^{\varnothing}_{\Lambda_n} - \psi^{\varnothing}_{[\Lambda_n]}| + |\psi^{\varnothing}_{[\Lambda_n]} - \psi^{\varnothing}_{D_k}| + |\psi^{\varnothing}_{D_k} - \psi|. \tag{3.4}$$

Fix $\epsilon > 0$. Since $\psi^{\varnothing}_{D_k} \to \psi$ when $k \to \infty$, there exists k_0, depending on β and ϵ, such that $|\psi^{\varnothing}_{D_k} - \psi| \le \epsilon/3$ for all $k \ge k_0$. We then compute $\psi^{\varnothing}_{[\Lambda_n]}$ by writing

$$\mathscr{H}^{\varnothing}_{[\Lambda_n]} = \sum_j \mathscr{H}^{\varnothing}_{D_k^{(j)}} + W_n,$$

where $|W_n| \le \beta \frac{|[\Lambda_n]|}{|D_k|} d(2^k)^{d-1} = \beta d 2^{-k} |[\Lambda_n]|$. Therefore, there exists k_1 (also depending on β and ϵ) such that

$$|\psi^{\varnothing}_{[\Lambda_n]} - \psi^{\varnothing}_{D_k}| \le \beta d 2^{-k} < \epsilon/3,$$

for all $k \ge k_1$. Let us then fix $k \ge \max\{k_0, k_1\}$. Let us write $\Delta_n \stackrel{\text{def}}{=} [\Lambda_n] \setminus \Lambda_n$. We observe that

$$\left|\mathscr{H}^{\varnothing}_{\Lambda_n} - \mathscr{H}^{\varnothing}_{[\Lambda_n]}\right| \le (2d\beta + |h|)\,|\Delta_n|.$$

Therefore,

$$\begin{aligned}\mathbf{Z}^{\varnothing}_{[\Lambda_n]} = \sum_{\omega\in\Omega_{[\Lambda_n]}} e^{-\mathscr{H}^{\varnothing}_{[\Lambda_n]}(\omega)} &\le \sum_{\omega\in\Omega_{\Lambda_n}} e^{-\mathscr{H}^{\varnothing}_{\Lambda_n}(\omega)} \sum_{\omega'\in\Omega_{\Delta_n}} e^{(2d\beta+|h|)\,|\Delta_n|} \\ &= e^{(2d\beta+|h|+\log 2)\,|\Delta_n|}\,\mathbf{Z}^{\varnothing}_{\Lambda_n}.\end{aligned}$$

Proceeding similarly to get a lower bound and observing that Δ_n contains at most $|\partial^{\text{in}}\Lambda_n||D_k|$ vertices, this yields

$$\left|\log \mathbf{Z}^{\varnothing}_{\Lambda_n} - \log \mathbf{Z}^{\varnothing}_{[\Lambda_n]}\right| \le |\partial^{\text{in}}\Lambda_n||D_k|\bigl(2d\beta + |h| + \log 2\bigr). \tag{3.5}$$

Since

$$1 \le \frac{|[\Lambda_n]|}{|\Lambda_n|} \le 1 + \frac{|\partial^{\text{in}}\Lambda_n||D_k|}{|\Lambda_n|}$$

and since $\psi^{\varnothing}_{\Lambda}$ is uniformly bounded (for example, by $2d\beta + |h| + \log 2$), it follows from (3.3) and (3.5) that

$$|\psi^{\varnothing}_{\Lambda_n} - \psi^{\varnothing}_{[\Lambda_n]}| \le \epsilon/3,$$

for all n large enough. Combining all these estimates, we conclude from (3.4) that, when n is sufficiently large,

$$|\psi^{\varnothing}_{\Lambda_n} - \psi| \le \epsilon.$$

(An alternative proof of convergence, using a subadditivity argument, is proposed in Exercise 3.3.)

▶ *Independence of boundary condition.* Let $\Lambda \Subset \mathbb{Z}^d$, $\eta \in \Omega$ and $\omega \in \Omega_\Lambda$. Denote by ω' the configuration in Ω^{η}_{Λ} coinciding with ω inside Λ. Then, $|\mathscr{H}_\Lambda(\omega') - \mathscr{H}^{\varnothing}_{\Lambda}(\omega)| \le 2d\beta|\partial^{\text{in}}\Lambda|$. This observation implies that

$$e^{-\beta 2d|\partial^{\text{in}}\Lambda|}\mathbf{Z}^{\varnothing}_{\Lambda} \le \mathbf{Z}^{\eta}_{\Lambda} \le e^{\beta 2d|\partial^{\text{in}}\Lambda|}\mathbf{Z}^{\varnothing}_{\Lambda}.$$

Applying this to each Λ_n and using (3.3) shows that $\lim_{\Lambda_n \Uparrow \mathbb{Z}^d} \psi^{\eta}_{\Lambda_n}$ exists and coincides with ψ. A completely similar argument, comparing $\mathbf{Z}^{\varnothing}_{V_n}$ and $\mathbf{Z}^{\text{per}}_{V_n}$, shows that $\lim_{n\to\infty} \psi^{\text{per}}_{V_n} = \psi$.

▶ *Convexity.* Since $(\beta, h) \mapsto \psi^{\#}_{\Lambda}(\beta, h)$ is convex (Lemma 3.5), its limit $\Lambda \Uparrow \mathbb{Z}^d$ is also convex (Exercise B.3).

▶ *Symmetry.* The fact that $h \mapsto \psi(\beta, h)$ is even is a direct consequence of the above and Exercise 3.2. □

The following exercise provides an alternative proof for the existence of the pressure (along a specific sequence of boxes), using a subadditivity argument. [1]

Exercise 3.3. *Let $\mathscr{R}$ be the set of all parallelepipeds of $\mathbb{Z}^d$, that is sets of the form* $\Lambda = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \cap \mathbb{Z}^d$.

1. *By writing $\sigma_i\sigma_j = (\sigma_i\sigma_j - 1) + 1$, express the Hamiltonian as $\mathscr{H}^{\varnothing}_{\Lambda} = \widetilde{\mathscr{H}}^{\varnothing}_{\Lambda} - \beta|\mathscr{E}_\Lambda|$, and observe that, for any disjoint sets $\Lambda_1, \Lambda_2 \Subset \mathbb{Z}^d$,*

$$\widetilde{\mathscr{H}}^{\varnothing}_{\Lambda_1\cup\Lambda_2} \ge \widetilde{\mathscr{H}}^{\varnothing}_{\Lambda_1} + \widetilde{\mathscr{H}}^{\varnothing}_{\Lambda_2}.$$

   Conclude that

$$\widetilde{\mathbf{Z}}^{\varnothing}_{\Lambda_1\cup\Lambda_2} \le \widetilde{\mathbf{Z}}^{\varnothing}_{\Lambda_1}\widetilde{\mathbf{Z}}^{\varnothing}_{\Lambda_2}. \tag{3.6}$$

2. *Use* (3.6) *and Lemma B.6 to show existence of* $\lim_{n\to\infty} \frac{1}{|\Lambda_n|}\log\widetilde{\mathbf{Z}}^{\varnothing}_{\Lambda_n}$ *along any sequence $\Lambda_n \uparrow \mathbb{Z}^d$ with $\Lambda_n \in \mathscr{R}$ for all n.*

3.2.3 Magnetization

As we already emphasized in the previous chapters, another quantity of central importance is the **magnetization density** in $\Lambda \Subset \mathbb{Z}^d$, which is the random variable

$$m_\Lambda \stackrel{\text{def}}{=} \frac{1}{|\Lambda|} M_\Lambda,$$

where $M_\Lambda \stackrel{\text{def}}{=} \sum_{i\in\Lambda} \sigma_i$ is the **total magnetization**. We also define, for any $\Lambda \Subset \mathbb{Z}^d$,

$$m^{\#}_{\Lambda}(\beta, h) \stackrel{\text{def}}{=} \langle m_\Lambda \rangle^{\#}_{\Lambda;\beta,h}.$$

As can be easily checked,

$$m^{\#}_{\Lambda}(\beta,h) = \frac{\partial \psi^{\#}_{\Lambda}}{\partial h}(\beta,h)\,. \tag{3.7}$$

Exercise 3.4. *Check that, more generally, the* ***cumulant generating function*** *associated to* M_Λ *(see Appendix B.8.3) can be expressed as*

$$\log\langle e^{tM_\Lambda}\rangle^{\#}_{\Lambda;\beta,h} = |\Lambda|\big(\psi^{\#}_{\Lambda}(\beta,h+t) - \psi^{\#}_{\Lambda}(\beta,h)\big)\,.$$

Deduce that the r *th cumulant of* M_Λ *is given by*

$$c_r(M_\Lambda) = |\Lambda|\,\frac{\partial^r \psi^{\#}_{\Lambda}}{\partial h^r}(\beta,h)\,.$$

The observation made in the previous exercise explains the important role played by the pressure, a fact that might surprise a reader with little familiarity with physics; after all, the partition function is just a normalizing factor. Indeed, we explain in Appendix B.8.3 that the cumulant generating function of a random variable encodes all the information about its distribution. In view of the central importance of the magnetization in characterizing the phase transition, as explained in Chapters 1 and 2, the pressure should hold precious information about the occurrence of a phase transition in the model. ◇

It will turn out to be important to determine whether (3.7) still holds in the thermodynamic limit. There are really two issues here: on the one hand, one has to address the existence of $\lim_{\Lambda\Uparrow\mathbb{Z}^d} \frac{\partial\psi^{\#}_{\Lambda}}{\partial h}(\beta,h)$ and whether the limit depends on the chosen boundary condition; on the other hand, there is also the problem of interchanging the thermodynamic limit and the differentiation with respect to h, that is, to verify whether it is true that

$$\lim_{\Lambda\Uparrow\mathbb{Z}^d} \frac{\partial \psi^{\#}_{\Lambda}}{\partial h} \stackrel{?}{=} \frac{\partial}{\partial h} \lim_{\Lambda\Uparrow\mathbb{Z}^d} \psi^{\#}_{\Lambda} = \frac{\partial \psi}{\partial h}\,.$$

These issues are intimately related to the differentiability of the pressure as a function of h. This is a delicate matter, which will be investigated in Section 3.7. Nevertheless, partial answers can already be deduced from the convexity properties of the pressure.

For instance, the one-sided derivatives of $h \mapsto \psi(\beta,h)$,

$$\frac{\partial\psi}{\partial h^-}(\beta,h) \stackrel{\text{def}}{=} \lim_{h'\uparrow h} \frac{\psi(\beta,h') - \psi(\beta,h)}{h'-h}\,, \qquad \frac{\partial\psi}{\partial h^+}(\beta,h) \stackrel{\text{def}}{=} \lim_{h'\downarrow h} \frac{\psi(\beta,h') - \psi(\beta,h)}{h'-h}\,,$$

exist everywhere (by item 1 of Theorem B.12) and are respectively left- and right-continuous (by item 5). Of course, the pressure will be differentiable with respect to h if and only if these two one-sided derivatives coincide. It is thus natural to introduce, for each β, the set

$$\begin{aligned}\mathfrak{B}_\beta &\stackrel{\text{def}}{=} \{h\in\mathbb{R} : \psi(\beta,\cdot) \text{ is not differentiable at } h\}\\ &= \{h\in\mathbb{R} : \tfrac{\partial\psi}{\partial h^-}(\beta,h) \neq \tfrac{\partial\psi}{\partial h^+}(\beta,h)\}\,.\end{aligned}$$

It follows from item 6 of Theorem B.12 that, for each β, the set $\mathfrak{B}_\beta$ is at most countable. On the complement of this set, one can answer the question raised above.

Corollary 3.7. *For all $h \notin \mathfrak{B}_\beta$, the* ***average magnetization density***

$$m(\beta,h) \stackrel{\text{def}}{=} \lim_{\Lambda\Uparrow\mathbb{Z}^d} m_\Lambda^{\#}(\beta,h)$$

is well defined, independent of the sequence $\Lambda \Uparrow \mathbb{Z}^d$ and of the boundary condition and satisfies

$$m(\beta,h) = \frac{\partial\psi}{\partial h}(\beta,h)\,. \tag{3.8}$$

Moreover, the function $h \mapsto m(\beta,h)$ is non-decreasing on $\mathbb{R}\setminus\mathfrak{B}_\beta$ and is continuous at every $h \notin \mathfrak{B}_\beta$. It is however discontinuous at each $h \in \mathfrak{B}_\beta$: for any $h \in \mathfrak{B}_\beta$,

$$\lim_{h'\downarrow h} m(\beta,h') = \tfrac{\partial\psi}{\partial h^+}(\beta,h)\,, \quad \lim_{h'\uparrow h} m(\beta,h') = \tfrac{\partial\psi}{\partial h^-}(\beta,h)\,. \tag{3.9}$$

In particular, the ***spontaneous magnetization***

$$m^*(\beta) \stackrel{\text{def}}{=} \lim_{h\downarrow 0} m(\beta,h)$$

is always well defined.

Proof. When $h \notin \mathfrak{B}_\beta$,

$$\frac{\partial\psi}{\partial h}(\beta,h) = \frac{\partial}{\partial h}\lim_{\Lambda\Uparrow\mathbb{Z}^d}\psi_\Lambda^{\#}(\beta,h) = \lim_{\Lambda\Uparrow\mathbb{Z}^d}\frac{\partial}{\partial h}\psi_\Lambda^{\#}(\beta,h) = \lim_{\Lambda\Uparrow\mathbb{Z}^d} m_\Lambda^{\#}(\beta,h)\,,$$

which proves (3.8), the existence of the thermodynamic limit of the magnetization density and the fact that it depends neither on the boundary condition nor on the sequence of volumes. Above, the second equality follows from item 7 of Theorem B.12 and the third one from (3.7).

The monotonicity and continuity of $h \mapsto m(\beta,h)$ on $\mathbb{R}\setminus\mathfrak{B}_\beta$ follow from (3.8) and items 4 and 5 of Theorem B.12.

Suppose now that $h \in \mathfrak{B}_\beta$ and let $(h_k)_{k\geq 1}$ be an arbitrary sequence in $\mathbb{R}\setminus\mathfrak{B}_\beta$ such that $h_k \downarrow h$ (there are always such sequences, since $\mathfrak{B}_\beta$ is at most countable). By (3.8), $\frac{\partial\psi}{\partial h^+}(\beta,h_k) = m(\beta,h_k)$ for all k. The claim (3.9) thus follows from (3.8) and item 5 of Theorem B.12. □

3.2.4 A first definition of phase transition

The above discussion shows that the average magnetization density is discontinuous precisely when the pressure is not differentiable in h. This leads to the following

Definition 3.8. *The pressure ψ exhibits a* ***first-order phase transition*** *at (β,h) if $h \mapsto \psi(\beta,h)$ fails to be differentiable at that point.*

Later, we will introduce another notion of first-order phase transition, of a more probabilistic nature. Determining whether phase transitions occur or not, and at which values of the parameters, is one of the main objectives of this chapter.

3.3 The one-dimensional Ising model

Before pursuing with the general case, we briefly discuss the one-dimensional Ising model, for which explicit computations are possible.

Theorem 3.9. ($d = 1$) *For all $\beta \ge 0$ and all $h \in \mathbb{R}$, the pressure $\psi(\beta, h)$ of the one-dimensional Ising model is given by*

$$\psi(\beta, h) = \log\Big\{e^{\beta}\cosh(h) + \sqrt{e^{2\beta}\cosh^2(h) - 2\sinh(2\beta)}\Big\}. \tag{3.10}$$

The explicit expression (3.10) shows that $h \mapsto \psi(\beta, h)$ is differentiable (real-analytic in fact) everywhere, for all $\beta \ge 0$, thus showing that $\mathfrak{B}_\beta = \varnothing$ when $d = 1$.

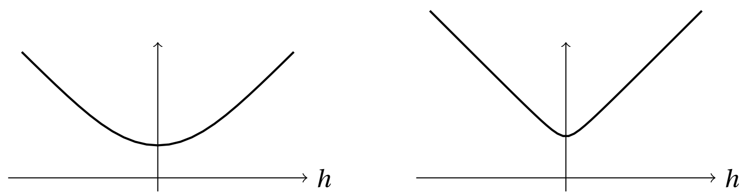


Figure 3.4: The pressure $h \mapsto \psi(\beta, h)$ of the one-dimensional Ising model, analytic in h at all temperature ($\beta = 0.8$ on the left, $\beta = 2$ on the right).

Consequently, as seen in Corollary 3.7, the average magnetization density $m(\beta, h)$ is given by

$$m(\beta, h) = \frac{\partial \psi}{\partial h}(\beta, h), \qquad \forall h \in \mathbb{R}.$$

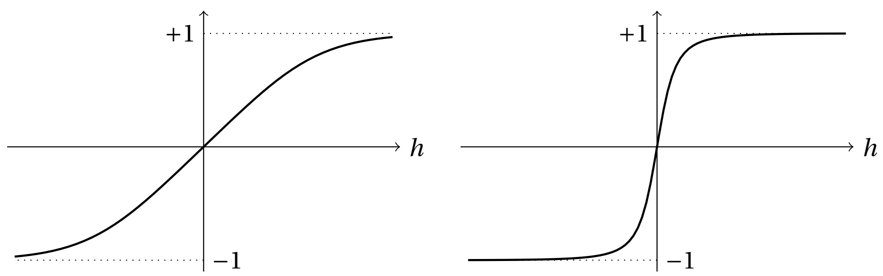


Figure 3.5: The average magnetization density $m(\beta, h)$ of the one-dimensional Ising model (for the same values of β as in Figure 3.4).

Since $h \mapsto \psi(\beta, h)$ is analytic, its derivative $h \mapsto m(\beta, h)$ is also analytic, in particular continuous. Therefore, $m^*(\beta) = \lim_{h \downarrow 0} m(\beta, h) = m(\beta, 0)$. But, since (see Exercise 3.2) $\psi(\beta, h) = \psi(\beta, -h)$, we get $\frac{\partial \psi}{\partial h}(\beta, 0) = 0$. This shows that the spontaneous magnetization of the one-dimensional Ising model is zero at all temperatures:

$$m^*(\beta) = 0, \qquad \forall \beta > 0.$$

In particular, the model exhibits *paramagnetic* behavior at all non-zero temperatures (remember the discussion in Section 1.4.3). We will provide an alternative proof of this fact in Section 3.7.3.

Only in the limit $\beta \to \infty$ does $\psi(\beta, h)$ become non-differentiable at $h = 0$, as seen in the following exercise.

Exercise 3.5. *Using* (3.10), *compute* $m(\beta, h)$. *Check that*

$$\lim_{h\to\pm\infty} m(\beta, h) = \pm 1, \quad \forall \beta \geq 0,$$

$$\lim_{\beta\to\infty} m(\beta, h) = \begin{cases} +1 & \text{if } h > 0, \\ 0 & \text{if } h = 0, \\ -1 & \text{if } h < 0. \end{cases}$$

Proof of Theorem 3.9: As seen in Theorem 3.6, the pressure is independent of the choice of boundary condition and of the sequence of volumes $\Lambda \Uparrow \mathbb{Z}$. The most convenient choice is to work on the torus $\mathbb{T}_n$, that is, to use $V_n = \{0, \dots, n-1\}$ with periodic boundary conditions; see Figure 3.1 (left). The advantage of this particular choice is that $\mathbf{Z}^{\text{per}}_{V_n;\beta,h}$ can be written as the trace of a 2×2 matrix. Indeed, writing $\omega_n \equiv \omega_0$,

$$\begin{aligned} \mathbf{Z}^{\text{per}}_{V_n;\beta,h} &= \sum_{\omega\in\Omega_{V_n}} e^{-\mathscr{H}^{\text{per}}_{V_n;\beta,h}(\omega)} \\ &= \sum_{\omega_0=\pm1} \cdots \sum_{\omega_{n-1}=\pm1} \prod_{i=0}^{n-1} e^{\beta\omega_i\omega_{i+1}+h\omega_i} \\ &= \sum_{\omega_0=\pm1} \cdots \sum_{\omega_{n-1}=\pm1} \prod_{i=0}^{n-1} A_{\omega_i,\omega_{i+1}}, \end{aligned}$$

where the numbers $A_{+,+} = e^{\beta+h}$, $A_{+,-} = e^{-\beta+h}$, $A_{-,+} = e^{-\beta-h}$ and $A_{-,-} = e^{\beta-h}$ can be arranged in the form of a matrix, called the **transfer matrix**:

$$A \stackrel{\text{def}}{=} \begin{pmatrix} e^{\beta+h} & e^{-\beta+h} \\ e^{-\beta-h} & e^{\beta-h} \end{pmatrix}. \tag{3.11}$$

The useful observation is that $\mathbf{Z}^{\text{per}}_{V_n;\beta,h}$ can now be interpreted as the trace of the nth power of A:

$$\mathbf{Z}^{\text{per}}_{V_n;\beta,h} = \sum_{\omega_0=\pm1} (A^n)_{\omega_0,\omega_0} = \text{Tr}(A^n).$$

A straightforward computation shows that the eigenvalues λ_+ and λ_- of A are given by

$$\lambda_\pm = e^\beta \cosh(h) \pm \sqrt{e^{2\beta}\cosh^2(h) - 2\sinh(2\beta)}.$$

Writing $A = BDB^{-1}$, with $D = \left(\begin{smallmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{smallmatrix}\right)$, and using the fact that $\text{Tr}(GH) = \text{Tr}(HG)$, we get

$$\mathbf{Z}^{\text{per}}_{V_n;\beta,h} = \text{Tr}(A^n) = \text{Tr}(BD^nB^{-1}) = \text{Tr}(D^n) = \lambda_+^n + \lambda_-^n.$$

Since $\lambda_+ > \lambda_-$, this gives $\psi(\beta, h) = \log \lambda_+$ and (3.10) is proved. (An interested reader with some familiarity with discrete-time, finite-state Markov chains can find some additional information on this topic in Section 3.10.4.) □

When $h = 0$, there exist several simple ways of computing the pressure of the one-dimensional Ising model: two are proposed in the following exercise and another one will be proposed in Exercise 3.26.

Exercise 3.6. *(Assuming $h = 0$.)*

1. *Configurations can be characterized by the collection of edges $\{i, i+1\}$ such that $\omega_i \neq \omega_{i+1}$. What is the contribution of a configuration with k such edges? Use that to compute the pressure.*

2. *Express the partition function in terms of the variables $(\omega_1, \tau_1, \dots, \tau_{n-1})$, where $\tau_i = \omega_{i-1}\omega_i$. Use this to compute the pressure.*

Hint: *since this does not affect the end result, one should choose a boundary condition that simplifies the analysis. We recommend using free boundary condition.*

With an explicit analytic expression for the pressure, we can extract information on the typical values of the magnetization density in large finite boxes. We will only consider the case $h = 0$; the extension to an arbitrary magnetic field is left as an exercise.

A consequence of the next theorem is that m_{Λ_n} concentrates on 0 under $\mu^{\#}_{\Lambda_n;\beta,0}$, as $n \to \infty$, for any type of boundary condition.

Theorem 3.10. *($d = 1$) Let $0 < \beta < \infty$ and consider any sequence $\Lambda_n \Uparrow \mathbb{Z}$, with an arbitrary boundary condition #. For all $\epsilon > 0$, there exists $c = c(\beta,\epsilon) > 0$ such that, for large enough n,*

$$\mu^{\#}_{\Lambda_n;\beta,0}\big(m_{\Lambda_n} \not\in (-\epsilon,\epsilon)\big) \le e^{-c|\Lambda_n|}. \tag{3.12}$$

Proof of Theorem 3.10: We start by writing

$$\mu^{\#}_{\Lambda_n;\beta,0}\big(m_{\Lambda_n} \not\in (-\epsilon,\epsilon)\big) = \mu^{\#}_{\Lambda_n;\beta,0}(m_{\Lambda_n} \ge \epsilon) + \mu^{\#}_{\Lambda_n;\beta,0}(m_{\Lambda_n} \le -\epsilon),$$

These two terms can be studied in the same way. The starting point is to use Chernov's Inequality (B.19): for all $h \ge 0$,

$$\mu^{\#}_{\Lambda_n;\beta,0}(m_{\Lambda_n} \ge \epsilon) \le e^{-h\epsilon|\Lambda_n|}\langle e^{hm_{\Lambda_n}|\Lambda_n|}\rangle^{\#}_{\Lambda_n;\beta,0}.$$

Since $\langle e^{hm_{\Lambda_n}|\Lambda_n|}\rangle^{\#}_{\Lambda_n;\beta,0} = \mathbf{Z}^{\#}_{\Lambda_n;\beta,h}/\mathbf{Z}^{\#}_{\Lambda_n;\beta,0}$, we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{|\Lambda_n|}\log\mu^{\#}_{\Lambda_n;\beta,0}(m_{\Lambda_n} \ge \epsilon) &\le \lim_{n\to\infty}\big(\psi^{\#}_{\Lambda_n}(\beta,h) - \psi^{\#}_{\Lambda_n}(\beta,0)\big) - h\epsilon \\ &= I_\beta(h) - h\epsilon,\end{aligned}$$

where $I_\beta(h) \stackrel{\text{def}}{=} \psi(\beta,h) - \psi(\beta,0)$. Since $h \ge 0$ was arbitrary, we can minimize over the latter:

$$\limsup_{n\to\infty} \frac{1}{|\Lambda_n|}\log\mu^{\#}_{\Lambda_n;\beta,0}(m_{\Lambda_n} \ge \epsilon) \le -\sup_{h\ge0}\{h\epsilon - I_\beta(h)\}. \tag{3.13}$$

In order to prove that $\mu^{\#}_{\Lambda_n;\beta}(m_{\Lambda_n} \ge \epsilon)$ decays exponentially fast in n, one must establish that $\sup_{h\ge0}\{h\epsilon - I_\beta(h)\} > 0$. Remember that the explicit expression for ψ provided by Theorem 3.9 is real-analytic in h. Moreover, $I_\beta(0) = 0$ and, if $I'_\beta = \frac{\partial}{\partial h} I_\beta$, then $I'_\beta(0) = 0$ and $I'_\beta(h) \to 1$ as $h \to \infty$, as was seen in Exercise 3.5. Therefore, for each $0 < \epsilon < 1$, there exists some $h_* > 0$, depending on ϵ and β, such that $\sup_{h\ge0}\{h\epsilon - I_\beta(h)\} = h_*\epsilon - I_\beta(h_*) > 0$ (see Figure 3.6). This proves (3.12). □

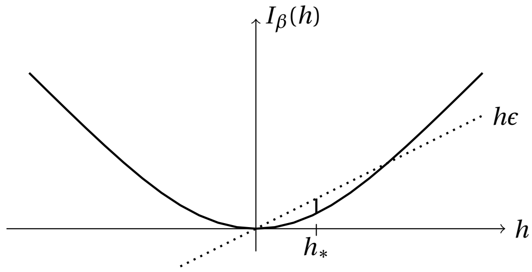


Figure 3.6: A picture showing the graphs of $h \mapsto I_\beta(h) = \psi(\beta, h) - \psi(\beta, 0)$ and $h \mapsto h\epsilon$, on which it is clear that $\sup_{h\geq 0}\{h\epsilon - I_\beta(h)\} > 0$ as soon as $\epsilon > 0$.

Exercise 3.7. *Proceeding as above, show that, under $\mu^{\#}_{\Lambda_n;\beta,h}$ with $h \neq 0$, m_{Λ_n} converges to $m(\beta, h)$ as $n \to \infty$ (in the same sense as in* (3.12)*), for any boundary condition.*

As explained above, the pressure contains a lot of information on the magnetization density. We will see in the following sections that smoothness of the pressure also guarantees *uniqueness of the infinite-volume Gibbs state*.

As we have seen in this section, explicitly computing the pressure yields useful information on the system. Unfortunately, computing the pressure becomes much more difficult, if at all possible, in higher dimensions. In fact, in spite of much effort, the only known results are for the two-dimensional Ising model with $h = 0$. In the latter case, Onsager determined, in a celebrated work, the explicit expression for the pressure:

$$\psi(\beta,0) = \log 2 + \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log\bigl\{(\cosh(2\beta))^2 - \sinh(2\beta)(\cos\theta_1 + \cos\theta_2)\bigr\}\,\mathrm{d}\theta_1\mathrm{d}\theta_2\,. \tag{3.14}$$

If we want to gain some understanding of the behavior of the Ising model on $\mathbb{Z}^d$, $d \geq 2$, other approaches are therefore required. This will be our main focus in the remainder of this chapter.

3.4 Infinite-volume Gibbs states

The pressure only provides information about the thermodynamical behavior of the system in large volumes. If one is interested in the statistical properties of general observables, such as the fluctuations of the magnetization density in a finite region or the correlations between far apart spins, one needs to understand the behavior of the Gibbs distribution $\mu^{\#}_{\Lambda;\beta,h}$ in large volumes.

One way of doing is to define infinite-volume Gibbs measures by taking some sequence $\Lambda_n \uparrow \mathbb{Z}^d$ and by considering the accumulation points (if any) of sequences of the type $(\mu^{\eta_n}_{\Lambda_n;\beta,h})_{n\geq 1}$. This is possible and will be done in detail in Chapter 6, by introducing a suitable notion of convergence for sequences of probability measures. Such an approach necessitates, however, rather abstract topological and measure-theoretic notions. In the present chapter, we avoid this, by following a more hands-on approach: a *state* (in infinite volume) will be identified with an assignment of an average value to each *local function*, that is, to each observable whose value only depends on finitely many spins.

Definition 3.11. *A function $f : \Omega \to \mathbb{R}$ is **local** if there exists $\Delta \Subset \mathbb{Z}^d$ such that $f(\omega) = f(\omega')$ as soon as ω and ω' coincide on Δ. The smallest*[1] *such set Δ is called the **support** of f and denoted by* $\mathrm{supp}(f)$.

For example, the value taken by the spin at the origin, σ_0, or the magnetization density in a set $\Lambda \Subset \mathbb{Z}^d$, $m_\Lambda = \frac{1}{|\Lambda|}\sum_{i\in\Lambda}\sigma_i$, are local functions with supports given respectively by $\{0\}$ and Λ.

Remark 3.12. In the sequel, we will occasionally make the following mild abuse of notation: if $f : \Omega \to \mathbb{R}$ is a local function and $\Delta \supset \mathrm{supp}(f)$, then, for any $\omega' \in \Omega_\Delta$, $f(\omega')$ is defined as the value of f evaluated at any configuration $\omega \in \Omega$ such that $\omega_i = \omega'_i$ for all $i \in \Delta$. (Clearly, that value does not depend on the choice of ω.) ⋄

Definition 3.13. *An **infinite-volume state** (or simply a **state**) is a mapping associating to each local function f a real number $\langle f\rangle$ and satisfying:*

Normalization: $\langle 1\rangle = 1$.

Positivity: If $f \geq 0$, then $\langle f\rangle \geq 0$.

Linearity: For any $\lambda \in \mathbb{R}$, $\langle f + \lambda g\rangle = \langle f\rangle + \lambda\langle g\rangle$.

*The number $\langle f\rangle$ is called the **average of f in the state** $\langle\cdot\rangle$.*

Definition 3.14. *Let $\Lambda_n \uparrow \mathbb{Z}^d$ and $(\#_n)_{n\geq 1}$ be a sequence of boundary conditions. The sequence of Gibbs distributions $(\mu^{\#_n}_{\Lambda_n;\beta,h})_{n\geq 1}$ is said to **converge to the state** $\langle\cdot\rangle$ if and only if*

$$\lim_{n\to\infty} \langle f\rangle^{\#_n}_{\Lambda_n;\beta,h} = \langle f\rangle\,,$$

*for every local function f. The state $\langle\cdot\rangle$ is then called a **Gibbs state (at (β, h))**.*

We simply write, as a shorthand,

$$\langle\cdot\rangle = \lim_{n\to\infty} \langle\cdot\rangle^{\#_n}_{\Lambda_n;\beta,h}$$

to indicate that $\langle\cdot\rangle^{\#_n}_{\Lambda_n;\beta,h}$ converges to $\langle\cdot\rangle$.

The above notion of convergence is natural. Indeed, from a thermodynamical perspective, it is expected that the properties of large systems at equilibrium should be well approximated by those of the corresponding infinite systems. In particular, finite-size effects, such as those resulting from the macroscopic shape of the system, should not affect local observations made far from the boundary of the system. The notion of convergence stated above corresponds precisely to a formalization of this principle, by saying that the measurement of a local quantity in a large system, corresponding to $\langle f\rangle^{\#_n}_{\Lambda_n;\beta,h}$, is well approximated by the corresponding measurement $\langle f\rangle$ in the infinite system. This is discussed in a more precise manner in Section 3.10.8. ⋄

Remark 3.15. The reader familiar with functional analysis will probably have noticed that, using the *Riesz–Markov–Kakutani representation theorem*, the average

[1] The reason one can speak about the *smallest* such set is the following observation: if a function f is characterized by $(\omega_i)_{i\in\Delta_1}$ and is also characterized by $(\omega_i)_{i\in\Delta_2}$, then it is characterized by $(\omega_i)_{i\in\Delta_1\cap\Delta_2}$.

$\langle f \rangle$ of a local function f in a state $\langle\cdot\rangle$ can always be seen as the expectation of f under some probability measure μ (on $\{+1,-1\}^{\mathbb{Z}^d}$):

$$\langle f \rangle = \int f \, d\mu .$$

We are mostly interested in states $\langle\cdot\rangle$ that can be constructed as limits of finite-volume Gibbs distributions: $\langle\cdot\rangle = \lim_{n\to\infty} \langle\cdot\rangle^{\eta_n}_{\Lambda_n;\beta,h}$. We will see that, in this case, the corresponding measure μ coincides with the weak limit of the probability measures $\mu^{\eta_n}_{\Lambda_n;\beta,h}$:

$$\mu^{\eta_n}_{\Lambda_n;\beta,h} \Rightarrow \mu .$$

This will be explained in Chapter 6, where the necessary framework for weak convergence of probability measures on $\{+1,-1\}^{\mathbb{Z}^d}$ will be introduced. ⋄

Since states are defined on the infinite lattice, it is natural to distinguish those that are *translation invariant*. The **translation** by $j \in \mathbb{Z}^d$, $\theta_j : \mathbb{Z}^d \to \mathbb{Z}^d$ is defined by

$$\theta_j i \stackrel{\text{def}}{=} i + j .$$

Translations can naturally be made to act on configurations: if $\omega \in \Omega$, then $\theta_j\omega$ is defined by

$$(\theta_j \omega)_i \stackrel{\text{def}}{=} \omega_{i-j} . \tag{3.15}$$

Definition 3.16. *A state $\langle\cdot\rangle$ is **translation invariant** if $\langle f \circ \theta_j \rangle = \langle f \rangle$ for every local function f and for all $j \in \mathbb{Z}^d$.*

The first important question is: *can Gibbs states be constructed for the Ising model with parameters (β, h)?* The following theorem shows that the constant-spin boundary conditions η^+ and η^- can be used to construct two states which will play a central role in the sequel.

Theorem 3.17. *Let $\beta \geq 0$ and $h \in \mathbb{R}$. Along any sequence $\Lambda_n \uparrow \mathbb{Z}^d$, the finite-volume Gibbs distributions with $+$- or $-$ boundary condition converge to infinite-volume Gibbs states:*

$$\langle\cdot\rangle^+_{\beta,h} = \lim_{n\to\infty} \langle\cdot\rangle^+_{\Lambda_n;\beta,h} , \qquad \langle\cdot\rangle^-_{\beta,h} = \lim_{n\to\infty} \langle\cdot\rangle^-_{\Lambda_n;\beta,h} . \tag{3.16}$$

The states $\langle\cdot\rangle^+_{\beta,h}$, $\langle\cdot\rangle^-_{\beta,h}$ do not depend on the sequence $(\Lambda_n)_{n\geq 1}$ and are both translation invariant.

The proof will be given later (on page 102), after introducing some important tools.

Remark 3.18. The previous theorem does not claim that $\langle\cdot\rangle^+_{\beta,h}$ and $\langle\cdot\rangle^-_{\beta,h}$ are distinct Gibbs states. Determining the set of values of the parameters β and h for which this is the case will be one of our main tasks in the remainder of this chapter. ⋄

More generally, one can prove, albeit in a non-constructive way, that any sequence of finite-volume Gibbs distributions admits converging subsequences.

Exercise 3.8. *Let $(\eta_n)_{n\geq 1}$ be a sequence of boundary conditions and $\Lambda_n \uparrow \mathbb{Z}^d$. Prove that there exists an increasing sequence $(n_k)_{k\geq 1}$ of integers and a Gibbs state $\langle\cdot\rangle$ such that*

$$\langle\cdot\rangle = \lim_{k\to\infty} \langle\cdot\rangle^{\eta_{n_k}}_{\Lambda_{n_k};\beta,h}$$

is well defined.

Revised version, August 22 2017
To be published by Cambridge University Press (2017)

Another explicit example using the free boundary condition will be considered in Exercise 3.16.

3.5 Two families of local functions.

The construction of Gibbs states consists in proving the existence of the limit

$$\lim_{n\to\infty} \langle f \rangle^{\eta_n}_{\Lambda_n;\beta,h}$$

for each local function f. Ideally, one would like to test convergence only on a restricted family of functions. The following lemma provides two particularly convenient such families, which will be especially well suited for the use of the correlation inequalities introduced in the next section. Define, for all $A \Subset \mathbb{Z}^d$,

$$\sigma_A \stackrel{\text{def}}{=} \prod_{j\in A} \sigma_j\,, \quad n_A \stackrel{\text{def}}{=} \prod_{j\in A} n_j\,,$$

where $n_j \stackrel{\text{def}}{=} \frac{1}{2}(1+\sigma_j)$ is the **occupation variable** at j.

Lemma 3.19. *Let f be local. There exist real coefficients $(\hat{f}_A)_{A\subset \text{supp}(f)}$ and $(\tilde{f}_A)_{A\subset\text{supp}(f)}$ such that both of the following representations hold:*

$$f = \sum_{A\subset\text{supp}(f)} \hat{f}_A \sigma_A\,, \qquad f = \sum_{A\subset\text{supp}(f)} \tilde{f}_A n_A\,.$$

Proof. The following orthogonality relation will be proved below: for all $B \Subset \mathbb{Z}^d$ and all configurations $\omega, \tilde{\omega}$,

$$2^{-|B|} \sum_{A\subset B} \sigma_A(\widetilde{\omega})\sigma_A(\omega) = \mathbf{1}_{\{\omega_i = \widetilde{\omega}_i,\, \forall i\in B\}}\,. \tag{3.17}$$

Applying (3.17) with $B = \text{supp}(f)$,

$$\begin{aligned} f(\omega) &= \sum_{\omega'\in\Omega_{\text{supp}(f)}} f(\omega')\,\mathbf{1}_{\{\omega_i=\omega'_i\ \forall i\in\text{supp}(f)\}} \\ &= \sum_{\omega'\in\Omega_{\text{supp}(f)}} f(\omega')\,2^{-|\text{supp}(f)|} \sum_{A\subset\text{supp}(f)} \sigma_A(\omega)\sigma_A(\omega') \\ &= \sum_{A\subset\text{supp}(f)} \Bigl\{ 2^{-|\text{supp}(f)|} \sum_{\omega'\in\Omega_{\text{supp}(f)}} f(\omega')\sigma_A(\omega') \Bigr\} \sigma_A(\omega)\,. \end{aligned}$$

This shows that the first identity holds with

$$\hat{f}_A = 2^{-|\text{supp}(f)|} \sum_{\omega'\in\Omega_{\text{supp}(f)}} f(\omega')\sigma_A(\omega')\,.$$

Since $\sigma_A = \prod_{i\in A}(2n_i - 1)$, the second identity follows from the first one.

We now prove (3.17). Let us first assume that $\omega_i = \widetilde{\omega}_i$, for all $i \in B$. In that case, $\sigma_A(\widetilde{\omega})\sigma_A(\omega) = \prod_{i\in A} \widetilde{\omega}_i\omega_i = 1$, since $\widetilde{\omega}_i\omega_i = \omega_i^2 = 1$ for all $i \in A \subset B$. This implies (3.17). Assume then that there exists $i \in B$ such that $\omega_i \neq \widetilde{\omega}_i$ (and thus $\omega_i \widetilde{\omega}_i = -1$). Then,

$$\begin{aligned}\sum_{A \subset B} \sigma_A(\widetilde{\omega})\sigma_A(\omega) &= \sum_{A \subset B\setminus\{i\}} \big(\sigma_A(\widetilde{\omega})\sigma_A(\omega) + \sigma_{A\cup\{i\}}(\widetilde{\omega})\sigma_{A\cup\{i\}}(\omega)\big)\\ &= \sum_{A \subset B\setminus\{i\}} (\sigma_A(\widetilde{\omega})\sigma_A(\omega) + \omega_i\widetilde{\omega}_i\sigma_A(\widetilde{\omega})\sigma_A(\omega))\\ &= \sum_{A \subset B\setminus\{i\}} \sigma_A(\widetilde{\omega})\sigma_A(\omega)\,(1+\omega_i\widetilde{\omega}_i) = 0\,. \qquad \square\end{aligned}$$

Thanks to the above lemma and to linearity, checking convergence of $(\langle f\rangle^{\eta_n}_{\Lambda_n;\beta,h})_{n\geq 1}$ for all local functions can now be reduced to showing convergence of $(\langle \sigma_A\rangle^{\eta_n}_{\Lambda_n;\beta,h})_{n\geq 1}$ or $(\langle n_A\rangle^{\eta_n}_{\Lambda_n;\beta,h})_{n\geq 1}$ for all finite $A \Subset \mathbb{Z}^d$. This task will be greatly simplified once we will have described some of the so-called *correlation inequalities* that hold for the Ising model.

3.6 Correlation inequalities

Correlation inequalities are one of the major tools in the mathematical analysis of the Ising model. We will use them to construct $\langle\cdot\rangle^+_{\beta,h}$ and $\langle\cdot\rangle^-_{\beta,h}$, and to study many other properties.

The Ising model enjoys many such inequalities, but we will restrict our attention to the two most prominent ones: the GKS and FKG inequalities. Since the proofs are not particularly enlightening, they are postponed to the end of the chapter, in Section 3.8.

3.6.1 The GKS inequalities.

As a motivation, consider the Ising model in a volume Λ, with + boundary condition. First, the ferromagnetic nature of the model makes it likely that the + boundary condition will favor a nonnegative magnetization inside the box, at least when $h \geq 0$. Therefore, if i is any point of Λ, it seems reasonable to expect that $h \geq 0$ implies

$$\langle \sigma_i\rangle^+_{\Lambda;\beta,h} \geq 0\,. \tag{3.18}$$

Similarly, knowing that the spin at some vertex j takes the value $+1$ should not decrease the probability of observing a + spin at another given vertex i, that is, one would expect that

$$\mu^+_{\Lambda;\beta,h}(\sigma_i = 1\,|\,\sigma_j = 1) \geq \mu^+_{\Lambda;\beta,h}(\sigma_i = 1)\,,$$

which can equivalently be written

$$\mu^+_{\Lambda;\beta,h}(\sigma_i = 1, \sigma_j = 1) \geq \mu^+_{\Lambda;\beta,h}(\sigma_i = 1)\,\mu^+_{\Lambda;\beta,h}(\sigma_j = 1)\,.$$

Since $\mathbf{1}_{\{\sigma_i = 1\}} = \frac{1}{2}(\sigma_i + 1)$, this can also be expressed as

$$\langle \sigma_i\sigma_j\rangle^+_{\Lambda;\beta,h} \geq \langle \sigma_i\rangle^+_{\Lambda;\beta,h}\langle \sigma_j\rangle^+_{\Lambda;\beta,h}\,. \tag{3.19}$$

This is equivalent to asking whether σ_i and σ_j are **positively correlated** under $\mu^+_{\Lambda;\beta,h}$.

Inequalities (3.18) and (3.19) are actually true, and will be particular instances of the *GKS inequalities* (named after Griffiths, Kelly and Sherman) which hold in a more general setting.

Namely, let $\mathbf{J} = (J_{ij})$ be a collection of *nonnegative* real numbers J_{ij} indexed by pairs $\{i,j\} \in \mathscr{E}^{\mathrm{b}}_\Lambda$. Let also $\mathbf{h} = (h_i)$ be a collection of real numbers indexed by vertices of Λ. We write $\mathbf{h} \geq 0$ as a shortcut for $h_i \geq 0$ for all $i \in \Lambda$. We then write, for $\omega \in \Omega^\eta_\Lambda$,

$$\mathscr{H}_{\Lambda;\mathbf{J},\mathbf{h}}(\omega) \stackrel{\text{def}}{=} -\sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda} J_{ij}\sigma_i(\omega)\sigma_j(\omega) - \sum_{i\in\Lambda} h_i\sigma_i(\omega). \tag{3.20}$$

We denote the corresponding finite-volume Gibbs distribution by $\mu^\eta_{\Lambda;\mathbf{J},\mathbf{h}}$. Of course, we recover $\mathscr{H}_{\Lambda;\beta,h}$ and $\mu^\eta_{\Lambda;\beta,h}$ by setting $J_{ij} = \beta$ for all $\{i,j\} \in \mathscr{E}^{\mathrm{b}}_\Lambda$ and $h_i = h$ for all $i \in \Lambda$.

The GKS inequalities are mostly restricted to +, free and periodic boundary conditions and to nonnegative magnetic fields. They deal with expectations and covariances of random variables of the type σ_A, which is precisely what is needed for the study of the thermodynamic limit.

Theorem 3.20 (GKS inequalities)**.** *Let* $\mathbf{J},\mathbf{h}$ *be as above and* $\Lambda \Subset \mathbb{Z}^d$. *Assume that* $\mathbf{h} \geq 0$. *Then, for all* $A, B \subset \Lambda$,

$$\langle\sigma_A\rangle^+_{\Lambda;\mathbf{J},\mathbf{h}} \geq 0, \tag{3.21}$$

$$\langle\sigma_A\sigma_B\rangle^+_{\Lambda;\mathbf{J},\mathbf{h}} \geq \langle\sigma_A\rangle^+_{\Lambda;\mathbf{J},\mathbf{h}}\langle\sigma_B\rangle^+_{\Lambda;\mathbf{J},\mathbf{h}}. \tag{3.22}$$

These inequalities remain valid for $\langle\cdot\rangle^\varnothing_{\Lambda;\mathbf{J},\mathbf{h}}$ *and* $\langle\cdot\rangle^{\mathrm{per}}_{\Lambda;\mathbf{J},\mathbf{h}}$.

Exercise 3.9. *Let* $A \subset \Lambda \Subset \mathbb{Z}^d$. *Under the assumptions of Theorem 3.20, prove that* $\langle\sigma_A\rangle^+_{\Lambda;\mathbf{J},\mathbf{h}}$ *is nondecreasing in both* $\mathbf{J}$ *and* $\mathbf{h}$.

3.6.2 The FKG inequality.

The FKG Inequality (named after Fortuin, Kasteleyn and Ginibre) states that increasing events are positively correlated.

The total order on the set $\{-1,1\}$ induces a **partial order** on Ω : $\omega \leq \omega'$ if and only if $\omega_i \leq \omega'_i$ for all $i \in \mathbb{Z}^d$. An event $E \subset \Omega$ is **increasing** if $\omega \in E$ and $\omega \leq \omega'$ implies $\omega' \in E$. If E and F are both increasing events depending on the spins inside Λ, then again, due to the ferromagnetic nature of the model, one can expect that the occurrence of an increasing event enhances the probability of another increasing event. That is, assuming that F has positive probability:

$$\mu^+_{\Lambda;\beta,h}(E\,|\,F) \geq \mu^+_{\Lambda;\beta,h}(E).$$

Multiplying by the probability of F, this inequality can be written as:

$$\mu^+_{\Lambda;\beta,h}(E\cap F) \geq \mu^+_{\Lambda;\beta,h}(E)\mu^+_{\Lambda;\beta,h}(F). \tag{3.23}$$

The precise result will be stated and proved in a more general setting, involving the expectation of nondecreasing local functions, of which $\mathbf{1}_E$ and $\mathbf{1}_F$ are particular instances.

A function $f : \Omega \to \mathbb{R}$ is **nondecreasing** if and only if $f(\omega) \leq f(\omega')$ for all $\omega \leq \omega'$.

Revised version, August 22 2017
To be published by Cambridge University Press (2017)

www.unige.ch/math/folks/velenik/smbook

Exercise 3.10. *Prove that the following functions are nondecreasing:* σ_i, n_i, n_A, $\sum_{i\in A} n_i - n_A$, *for any* $i \in \mathbb{Z}^d$, $A \Subset \mathbb{Z}^d$.

A particularly useful feature of the FKG inequality is its applicability for all possible boundary conditions and arbitrary (that is, not necessarily nonnegative) values of the magnetic field. They are also valid in the general setting presented in the last section, in which β and h are replaced by $\mathbf{J}$ and $\mathbf{h}$:

Theorem 3.21 (FKG inequality)**.** *Let* $\mathbf{J} = (J_{ij})_{i,j\in\mathbb{Z}^d}$ *be a collection of nonnegative real numbers and let* $\mathbf{h} = (h_i)_{i\in\mathbb{Z}^d}$ *be a collection of arbitrary real numbers. Let* $\Lambda \Subset \mathbb{Z}^d$ *and # be some arbitrary boundary condition. Then, for any pair of nondecreasing functions* f *and* g,

$$\langle fg\rangle^{\#}_{\Lambda;\mathbf{J},\mathbf{h}} \geq \langle f\rangle^{\#}_{\Lambda;\mathbf{J},\mathbf{h}} \langle g\rangle^{\#}_{\Lambda;\mathbf{J},\mathbf{h}} . \tag{3.24}$$

Inequality (3.23) follows by taking $J_{ij} = \beta$ and $h_i = h$, and $f = \mathbf{1}_E$, $g = \mathbf{1}_F$. Note also that $\langle fg\rangle^{\eta}_{\Lambda;\mathbf{J},\mathbf{h}} \leq \langle f\rangle^{\eta}_{\Lambda;\mathbf{J},\mathbf{h}} \langle g\rangle^{\eta}_{\Lambda;\mathbf{J},\mathbf{h}}$ whenever f is nondecreasing and g is nonincreasing (simply apply (3.24) to f and $-g$).

Actually, (3.24) *can be seen as a natural extension of the following elementary result: if* f *and* g *are two nondecreasing functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *and* μ *is a probability measure on* $\mathbb{R}$*, then*

$$\langle fg\rangle_\mu \geq \langle f\rangle_\mu \langle g\rangle_\mu .$$

Namely, it suffices to write

$$\langle fg\rangle_\mu - \langle f\rangle_\mu \langle g\rangle_\mu = \tfrac{1}{2}\int (f(x)-f(y))(g(x)-g(y))\mu(\mathrm{d}x)\mu(\mathrm{d}y),$$

and to observe that $f(x)-f(y)$ *and* $g(x)-g(y)$ *have the same sign, since* f *and* g *are both nondecreasing.* ◇

3.6.3 Consequences

Many useful properties of finite-volume Gibbs distributions can be derived from the correlation inequalities of the previous section. The first is exactly the ingredient that will be needed for the study of the thermodynamic limit:

Lemma 3.22. *Let* f *be a nondecreasing function and* $\Lambda_1 \subset \Lambda_2 \Subset \mathbb{Z}^d$*. Then, for any* $\beta \geq 0$ *and* $h \in \mathbb{R}$,

$$\langle f\rangle^{+}_{\Lambda_1;\beta,h} \geq \langle f\rangle^{+}_{\Lambda_2;\beta,h} . \tag{3.25}$$

The same statement holds for the $-$ *boundary condition and a nonincreasing function* f.

Before turning to the proof, we need a spatial Markov property satisfied by $\mu^{\eta}_{\Lambda;\beta,h}$.

Exercise 3.11. *Prove that, for all* $\Delta \subset \Lambda \Subset \mathbb{Z}^d$ *and all configurations* $\eta \in \Omega$ *and* $\omega' \in \Omega^{\eta}_{\Lambda}$,

$$\mu^{\eta}_{\Lambda;\beta,h}\big(\cdot \mid \sigma_i = \omega'_i, \forall i \in \Lambda\setminus\Delta\big) = \mu^{\omega'}_{\Delta;\beta,h}(\cdot) . \tag{3.26}$$

Revised version, August 22 2017
To be published by Cambridge University Press (2017)

The probability in the right-hand side of (3.26) *really only depends on* ω_i' *for* $i \in \partial^{\rm ex}\Delta$, *where* $\partial^{\rm ex}\Delta$ *is the* ***exterior boundary of*** $\boldsymbol{\Delta}$, *defined by*

$$\partial^{\rm ex}\Delta \stackrel{\rm def}{=} \{i \not\in \Delta : \exists j \in \Delta,\, j \sim i\}\,.$$

This implies that

$$\mu^\eta_{\Lambda;\beta,h}\big(A \mid \sigma_i = \omega_i',\, \forall i \in \Lambda \setminus \Delta\big) = \mu^\eta_{\Lambda;\beta,h}\big(A \mid \sigma_i = \omega_i',\, \forall i \in \partial^{\rm ex}\Delta\big)\,,$$

for all events A depending only on the spins located inside Δ. In this sense, (3.26) *is indeed a spatial Markov property.* ◇

Proof of Lemma 3.22: It follows from (3.26) that

$$\langle f\rangle^+_{\Lambda_1;\beta,h} = \langle f \mid \sigma_i = 1,\, \forall i \in \Lambda_2 \setminus \Lambda_1\rangle^+_{\Lambda_2;\beta,h}\,.$$

The indicator $\mathbf{1}_{\{\sigma_i=1,\,\forall i\in\Lambda_2\setminus\Lambda_1\}}$ being a nondecreasing function, the FKG inequality implies that

$$\begin{aligned}\langle f\rangle^+_{\Lambda_1;\beta,h} &= \frac{\big\langle f\mathbf{1}_{\{\sigma_i=1,\,\forall i\in\Lambda_2\setminus\Lambda_1\}}\big\rangle^+_{\Lambda_2;\beta,h}}{\big\langle \mathbf{1}_{\{\sigma_i=1,\,\forall i\in\Lambda_2\setminus\Lambda_1\}}\big\rangle^+_{\Lambda_2;\beta,h}}\\ &\geq \frac{\langle f\rangle^+_{\Lambda_2;\beta,h}\big\langle \mathbf{1}_{\{\sigma_i=1,\,\forall i\in\Lambda_2\setminus\Lambda_1\}}\big\rangle^+_{\Lambda_2;\beta,h}}{\big\langle \mathbf{1}_{\{\sigma_i=1,\,\forall i\in\Lambda_2\setminus\Lambda_1\}}\big\rangle^+_{\Lambda_2;\beta,h}} = \langle f\rangle^+_{\Lambda_2;\beta,h}\,.\end{aligned}$$ □

Actually, some form of monotonicity with respect to the volume can also be established for the Gibbs distributions with free boundary condition:

Exercise 3.12. *Using the GKS inequalities, prove that, for all $\beta, h \geq 0$,*

$$\langle \sigma_A\rangle^+_{\Lambda_1;\beta,h} \geq \langle \sigma_A\rangle^+_{\Lambda_2;\beta,h}\,, \qquad \langle \sigma_A\rangle^\varnothing_{\Lambda_1;\beta,h} \leq \langle \sigma_A\rangle^\varnothing_{\Lambda_2;\beta,h}\,,$$

for all $A \subset \Lambda_1 \subset \Lambda_2 \Subset \mathbb{Z}^d$.

The next lemma shows that the Gibbs distributions with + and − boundary condition play an extremal role, in the sense that they maximally favor +, respectively −, spins.

Lemma 3.23. *Let f be an arbitrary nondecreasing function. Then, for any $\beta \geq 0$ and $h \in \mathbb{R}$,*

$$\langle f\rangle^-_{\Lambda;\beta,h} \leq \langle f\rangle^\eta_{\Lambda;\beta,h} \leq \langle f\rangle^+_{\Lambda;\beta,h}\,,$$

for any boundary condition $\eta \in \Omega$ and any $\Lambda \Subset \mathbb{Z}^d$. Similarly, if f is a local function with $\mathrm{supp}(f) \subset \Lambda$, *resp.* $\mathrm{supp}(f) \subset V_N$, *then*

$$\langle f\rangle^-_{\Lambda;\beta,h} \leq \langle f\rangle^\varnothing_{\Lambda;\beta,h} \leq \langle f\rangle^+_{\Lambda;\beta,h}\,,$$
$$\langle f\rangle^-_{V_{N-1};\beta,h} \leq \langle f\rangle^{\rm per}_{V_N;\beta,h} \leq \langle f\rangle^+_{V_{N-1};\beta,h}\,.$$

Proof. Let $I(\omega) = \exp\{\beta \sum_{\substack{i\in\Lambda, j\not\in\Lambda\\ i\sim j}} \omega_i(1-\eta_j)\}$. First, observe that

$$\sum_{\omega\in\Omega^+_\Lambda} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)} = \sum_{\omega\in\Omega^\eta_\Lambda} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)} I(\omega)\,,$$

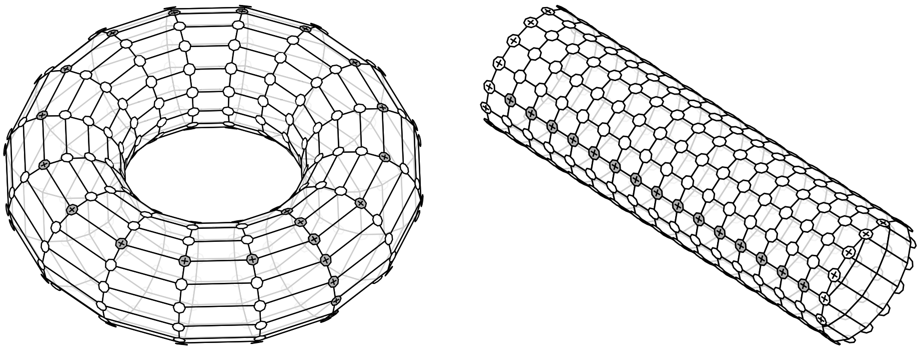


Figure 3.7: Left: The two-dimensional torus $\mathbb{T}_{16}$ with all spins along Σ_{16} forced to take the value +1. Right: opening the torus along the first "circle" of +1 yields an equivalent Ising model on a cylinder with + boundary condition and all spins forced to take the value +1 along a line. Further opening the cylinder along the line of frozen + spins yields an equivalent Ising model in the square $\{1,\dots,15\}^2$ with + boundary condition.

and, for any nondecreasing f,

$$\sum_{\omega\in\Omega_\Lambda^+} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)} f(\omega) \ge \sum_{\omega\in\Omega_\Lambda^\eta} e^{-\mathscr{H}_{\Lambda;\beta,h}(\omega)} I(\omega) f(\omega)\,.$$

(The inequality is a consequence of our not assuming that $\operatorname{supp}(f)\subset\Lambda$.) This implies that

$$\langle f\rangle^+_{\Lambda;\beta,h} = \frac{\sum_{\omega\in\Omega_\Lambda^+} e^{-\mathscr{H}_\Lambda(\omega)} f(\omega)}{\sum_{\omega\in\Omega_\Lambda^+} e^{-\mathscr{H}_\Lambda(\omega)}} \ge \frac{\sum_{\omega\in\Omega_\Lambda^\eta} e^{-\mathscr{H}_\Lambda(\omega)} I(\omega) f(\omega)}{\sum_{\omega\in\Omega_\Lambda^\eta} e^{-\mathscr{H}_\Lambda(\omega)} I(\omega)} = \frac{\langle If\rangle^\eta_{\Lambda;\beta,h}}{\langle I\rangle^\eta_{\Lambda;\beta,h}} \ge \langle f\rangle^\eta_{\Lambda;\beta,h}\,,$$

where we applied the FKG inequality in the last step, making use of the fact that the function I is nondecreasing.

The proof for the free boundary condition is identical, using the nondecreasing function $I(\omega) = \exp\{\beta\sum_{i\in\Lambda, j\notin\Lambda, i\sim j}\omega_i\}$.

Let us finally consider the Gibbs distribution with periodic boundary condition. In that case, we can argue as in the proof of Lemma 3.22, since, for any $\omega\in\Omega^+_{V_{N-1}}$ (considering $V_N=\{0,\dots,N\}^d$ as a subset of $\mathbb{Z}^d$),

$$\mu^{\mathrm{per}}_{V_N;\beta,h}\big(\omega|_{V_N}\;\big|\;\sigma_i = 1\ \forall i\in\Sigma_N\big) = \mu^+_{V_{N-1};\beta,h}(\omega)\,,$$

where $\Sigma_N \stackrel{\text{def}}{=} \big\{i=(i_1,\dots,i_d)\in V_N : \exists 1\le k\le d \text{ such that } i_k = 0\big\}$ (see Figure 3.7) and the **restriction** of a configuration $\omega\in\Omega$ to a subset $S\subset\mathbb{Z}^d$ is defined by

$$\omega|_S \stackrel{\text{def}}{=} (\omega_i)_{i\in S}\,. \qquad \square$$

Exercise 3.13. *Let $\eta,\omega\in\Omega$ be such that $\eta\le\omega$. Let f be a nondecreasing function. Show that, for any $\beta\ge 0$ and $h\in\mathbb{R}$,*

$$\langle f\rangle^\eta_{\Lambda;\beta,h} \le \langle f\rangle^\omega_{\Lambda;\beta,h}\,,$$

for any $\Lambda\Subset\mathbb{Z}^d$. Hint: *adapt the argument in the proof of Lemma 3.23.*

We can now prove existence and translation invariance of $\langle\cdot\rangle^+_{\beta,h}$ and $\langle\cdot\rangle^-_{\beta,h}$.

Proof of Theorem 3.17: We consider the + boundary condition. Let f be a local function. By Lemma 3.19 and linearity,

$$\langle f\rangle^+_{\Lambda_n;\beta,h} = \sum_{A\subset\mathrm{supp}(f)} \tilde f_A \langle n_A\rangle^+_{\Lambda_n;\beta,h}\,.$$

Since the functions n_A are nondecreasing, (3.25) implies that, for all A,

$$\langle n_A\rangle^+_{\Lambda_n;\beta,h} \geq \langle n_A\rangle^+_{\Lambda_{n+1};\beta,h}\,, \qquad \forall n\geq 1\,.$$

Being nonnegative, $\langle n_A\rangle^+_{\Lambda_n;\beta,h}$ thus converges as $n\to\infty$. It follows that $\langle f\rangle^+_{\Lambda_n;\beta,h}$ also has a limit, which we denote by

$$\langle f\rangle^+_{\beta,h} \stackrel{\text{def}}{=} \lim_{n\to\infty}\langle f\rangle^+_{\Lambda_n;\beta,h}\,.$$

Since it is obviously linear, positive and normalized, $\langle\cdot\rangle^+_{\beta,h}$ is a Gibbs state. We check now that it does not depend on the sequence $\Lambda_n\uparrow\mathbb{Z}^d$. Let $\Lambda^1_n\uparrow\mathbb{Z}^d$ and $\Lambda^2_n\uparrow\mathbb{Z}^d$ be two such sequences, and let us denote by $\langle\cdot\rangle^{+,1}_{\beta,h}$ and $\langle\cdot\rangle^{+,2}_{\beta,h}$ the corresponding limits. Since $\Lambda^1_n\uparrow\mathbb{Z}^d$ and $\Lambda^2_n\uparrow\mathbb{Z}^d$, we can always find a sequence $(\Delta_n)_{n\geq1}$ such that, for all $k\geq 1$,

$$\Delta_{2k-1}\in\{\Lambda^1_n : n\geq 1\}\,,\quad \Delta_{2k}\in\{\Lambda^2_n : n\geq 1\}\,,\quad \Delta_k\subsetneq\Delta_{k+1}\,.$$

Of course, $\Delta_n\uparrow\mathbb{Z}^d$. Our previous considerations thus imply that $\lim_{k\to\infty}\langle f\rangle^+_{\Delta_k;\beta,h}$ exists, for every local function f. Moreover, since $(\langle f\rangle^+_{\Delta_{2k-1};\beta,h})_{k\geq1}$ is a subsequence of $(\langle f\rangle^+_{\Lambda^1_n;\beta,h})_{n\geq1}$ and $(\langle f\rangle^+_{\Delta_{2k};\beta,h})_{k\geq1}$ is a subsequence of $(\langle f\rangle^+_{\Lambda^2_n;\beta,h})_{n\geq1}$, we conclude that

$$\lim_{n\to\infty}\langle f\rangle^+_{\Lambda^1_n;\beta,h} = \lim_{k\to\infty}\langle f\rangle^+_{\Delta_k;\beta,h} = \lim_{n\to\infty}\langle f\rangle^+_{\Lambda^2_n;\beta,h}\,,$$

for all local functions f. This shows that the state $\langle\cdot\rangle^+_{\beta,h}$ does not depend on the choice of the sequence $(\Lambda_n)_{n\geq1}$.

We still have to prove translation invariance. Let again f be a local function. For all $j\in\mathbb{Z}^d$, $f\circ\theta_j$ is also a local function and $\theta_{-j}\Lambda_n\uparrow\mathbb{Z}^d$ ($\theta_i\Lambda\stackrel{\text{def}}{=}\Lambda+i$). We thus have

$$\langle f\rangle^+_{\Lambda_n;\beta,h}\to\langle f\rangle^+_{\beta,h} \qquad\text{and}\qquad \langle f\circ\theta_j\rangle^+_{\theta_{-j}\Lambda_n;\beta,h}\to\langle f\circ\theta_j\rangle^+_{\beta,h}\,.$$

The conclusion follows, since $\langle f\circ\theta_j\rangle^+_{\theta_{-j}\Lambda_n;\beta,h} = \langle f\rangle^+_{\Lambda_n;\beta,h}$ (see Figure 3.8). □

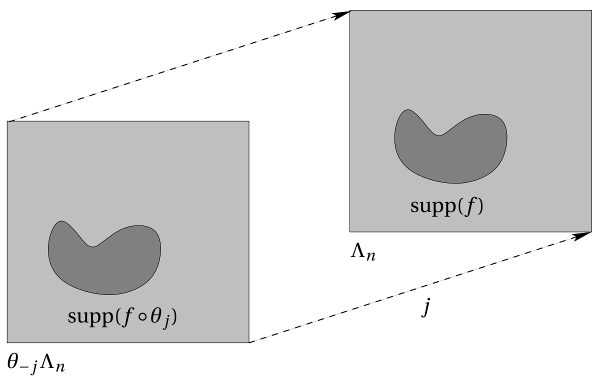


Figure 3.8: Proof of invariance under translations.

Exercise 3.14. *Prove that* $\langle\cdot\rangle^+_{\beta,h}$ *and* $\langle\cdot\rangle^-_{\beta,h}$ *are also invariant under lattice rotations and reflections of* $\mathbb{Z}^d$.

Exercise 3.15. *Let* $\beta \geq 0$ *and* $h \in \mathbb{R}$. *Show that* $\langle\cdot\rangle^+_{\beta,h}$ *has* ***short-range correlations****, in the sense that, for all local functions* f *and* g,

$$\lim_{\|i\|_1\to\infty} \langle f\cdot(g\circ\theta_i)\rangle^+_{\beta,h} = \langle f\rangle^+_{\beta,h}\langle g\rangle^+_{\beta,h}.$$

Hint: *Use the FKG inequality to prove first the result with* $f = n_A$ *and* $g = n_B$ *for arbitrary* $A, B \Subset \mathbb{Z}^d$.

With similar arguments, one can also construct Gibbs states using the free boundary condition:

Exercise 3.16. *Prove that, for all* $\beta \geq 0$, $h \in \mathbb{R}$ *and any sequence* $\Lambda_n \uparrow \mathbb{Z}^d$, *the sequence* $(\langle\cdot\rangle^\varnothing_{\Lambda_n;\beta,h})_{n\geq 1}$ *converges to a Gibbs state* $\langle\cdot\rangle^\varnothing_{\beta,h}$, *independent of the sequence* $(\Lambda_n)_{n\geq 1}$ *chosen. Show that* $\langle\cdot\rangle^\varnothing_{\beta,h}$ *is translation invariant.*

3.7 Phase Diagram

Now that we have seen that infinite-volume Gibbs states for a pair (β, h) can be constructed rigorously in various ways (for example, using + or − boundary conditions), the next problem is to determine whether these are the same Gibbs states, or whether there exist some values of the temperature and magnetic field for which the influence of the boundary condition survives in the thermodynamic limit, leading to multiple Gibbs states.

The answer to this question will be given in the next sections: it will depend on the dimension d and on the values of β and h. Contrarily to what often happens in mathematics, the lack of uniqueness is not a defect of this approach, but is actually one of its main features: lack of uniqueness means that providing a complete microscopic description of the system (that is, the set of configurations and the Hamiltonian) as well as fixing all the relevant thermodynamic parameters (β and h) is not sufficient to completely determine the macroscopic behavior of the system.

Definition 3.24. *If at least two distinct Gibbs states can be constructed for a pair* (β, h), *we say that there is a* ***first-order phase transition at*** $(\boldsymbol{\beta}, \boldsymbol{h})$.

Later in this chapter (see Theorem 3.34), we will relate this *probabilistic* definition of a first-order phase transition to the *analytic* one associated to the pressure (Definition 3.8).

We can now turn to the main result of this chapter, which establishes the *phase diagram* of the Ising model, that is, the determination for each pair (β, h) of whether there is a unique or multiple Gibbs states. We gather the corresponding claims in the form of a theorem, the proof of which will be given in the remainder of the chapter (see Figure 3.9).

Theorem 3.25. *1. In any $d \geq 1$, when $h \neq 0$, there is a unique Gibbs state for all values of $\beta \in \mathbb{R}_{\geq 0}$.*

2. In $d = 1$, there is a unique Gibbs state at each $(\beta, h) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$.

3. When $h = 0$ and $d \geq 2$, there exists $\beta_c = \beta_c(d) \in (0, \infty)$ such that:

- *when $\beta < \beta_c$, the Gibbs state at $(\beta, 0)$ is unique,*
- *when $\beta > \beta_c$, a first-order phase transition occurs at $(\beta, 0)$:*

$$\langle\cdot\rangle^+_{\beta,0} \neq \langle\cdot\rangle^-_{\beta,0}.$$

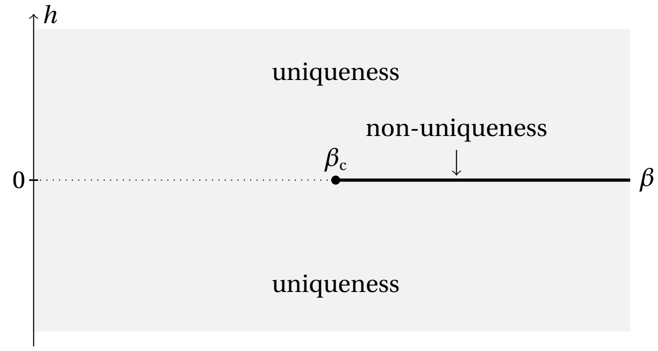


Figure 3.9: The phase diagram of the Ising model in $d \geq 2$. The line $\{(\beta, 0) : \beta > \beta_c\}$ is called the **coexistence line**. This diagram should be compared with the simulations of Figure 1.9.

The proof of Theorem 3.25 is quite long and is spread over several sections. The first item will be proved in Section 3.7.4. The second item was already proved in Section 3.3 (once the results there are combined with Theorem 3.34) and will be given an alternative proof in Section 3.7.3. The proof of the third item has two parts: the proof that $\beta_c < \infty$ is done in Section 3.7.2, while the proof that $\beta_c > 0$ is done in Section 3.7.3.

Remark 3.26. It can be proved that uniqueness holds also at $(\beta_c, 0)$, when $d \geq 2$, but the argument is beyond the scope of this book. [2] The phase transition occurring as β crosses β_c (at $h = 0$) is thus continuous. ⋄

Remark 3.27. Although the above theorem claims the existence of at least two distinct Gibbs states when $d \geq 2$, $h = 0$ and $\beta > \beta_c$, it does not describe the *structure* of the set of Gibbs states associated to those values of (β, h). This is a much more difficult problem, to which we will return in Section 3.10.8. ⋄

3.7.1 Two criteria for (non)-uniqueness

In this subsection, we establish a link between uniqueness of the Gibbs state, the average magnetization density and differentiability of the pressure. We use these quantities to formulate several equivalent characterizations of uniqueness of the Gibbs state, which play a crucial role in our determination of the phase diagram.

Moreover, the second criterion provides the rigorous link between the analytic and probabilistic definitions of first-order phase transition introduced earlier.

A first characterization of uniqueness

The major role played by the states $\langle\cdot\rangle^+_{\beta,h}$ and $\langle\cdot\rangle^-_{\beta,h}$ is made clear by the following result.

Theorem 3.28. *Let $(\beta,h)\in\mathbb{R}_{\geq 0}\times\mathbb{R}$. The following statements are equivalent:*

1. *There is a unique Gibbs state at (β,h).*
2. $\langle\cdot\rangle^+_{\beta,h}=\langle\cdot\rangle^-_{\beta,h}$.
3. $\langle\sigma_0\rangle^+_{\beta,h}=\langle\sigma_0\rangle^-_{\beta,h}$.

Proof. The implications 1⇒2⇒3 are trivial. Let us prove that 3⇒2. Take $\Lambda_n\uparrow\mathbb{Z}^d$ and $A\Subset\mathbb{Z}^d$. Since $\sum_{i\in A}n_i-n_A$ is nondecreasing (Exercise 3.10), Lemma 3.23 implies that, for all k,

$$\Big\langle\sum_{i\in A}n_i-n_A\Big\rangle^-_{\Lambda_k;\beta,h}\leq\Big\langle\sum_{i\in A}n_i-n_A\Big\rangle^+_{\Lambda_k;\beta,h}.$$

Using linearity, letting $k\to\infty$ and rearranging, we get

$$\sum_{i\in A}\big(\langle n_i\rangle^+_{\beta,h}-\langle n_i\rangle^-_{\beta,h}\big)\geq\langle n_A\rangle^+_{\beta,h}-\langle n_A\rangle^-_{\beta,h}.$$

If 3 holds, the left-hand side vanishes, since translation invariance then implies that

$$\langle n_i\rangle^+_{\beta,h}-\langle n_i\rangle^-_{\beta,h}=\langle n_0\rangle^+_{\beta,h}-\langle n_0\rangle^-_{\beta,h}=\tfrac{1}{2}\big(\langle\sigma_0\rangle^+_{\beta,h}-\langle\sigma_0\rangle^-_{\beta,h}\big)=0.$$

But $\langle n_A\rangle^+_{\beta,h}\geq\langle n_A\rangle^-_{\beta,h}$ (again by Lemma 3.23), and so $\langle n_A\rangle^+_{\beta,h}=\langle n_A\rangle^-_{\beta,h}$. Together with Lemma 3.19, this implies that $\langle f\rangle^+_{\beta,h}=\langle f\rangle^-_{\beta,h}$ for every local function f. Therefore, 2 holds.

It only remains to prove that 2⇒1. Lemma 3.23 implies that any Gibbs state at (β,h), say $\langle\cdot\rangle_{\beta,h}$, is such that $\langle n_A\rangle^-_{\beta,h}\leq\langle n_A\rangle_{\beta,h}\leq\langle n_A\rangle^+_{\beta,h}$. If 2 holds, this implies $\langle n_A\rangle^-_{\beta,h}=\langle n_A\rangle_{\beta,h}=\langle n_A\rangle^+_{\beta,h}$. By Lemma 3.19, this extends to all local functions and, therefore, $\langle\cdot\rangle^-_{\beta,h}=\langle\cdot\rangle_{\beta,h}=\langle\cdot\rangle^+_{\beta,h}$. We conclude that 1 holds. □

Some properties of the magnetization density

Remember that the average magnetization density in $\Lambda\Subset\mathbb{Z}^d$ with an arbitrary boundary condition # was defined by $m^\#_\Lambda(\beta,h)\stackrel{\text{def}}{=}\langle m_\Lambda\rangle^\#_{\Lambda;\beta,h}$. The uniqueness criterion developed in Theorem 3.28 is expressed in terms of the averages $\langle\sigma_0\rangle^+_{\beta,h}$ and $\langle\sigma_0\rangle^-_{\beta,h}$. It is natural to wonder whether these quantities are related to $m^+_\Lambda(\beta,h)$ and $m^-_\Lambda(\beta,h)$. The following result shows that they in fact coincide in the thermodynamic limit.

Proposition 3.29. *For any sequence $\Lambda \Uparrow \mathbb{Z}^d$, the limits*

$$m^+(\beta,h) \stackrel{\text{def}}{=} \lim_{\Lambda\Uparrow\mathbb{Z}^d} m^+_\Lambda(\beta,h)\,,\quad m^-(\beta,h) \stackrel{\text{def}}{=} \lim_{\Lambda\Uparrow\mathbb{Z}^d} m^-_\Lambda(\beta,h)$$

exist and

$$m^+(\beta,h) = \langle\sigma_0\rangle^+_{\beta,h}\,,\quad m^-(\beta,h) = \langle\sigma_0\rangle^-_{\beta,h}\,.$$

Moreover, $h \mapsto m^+(\beta,h)$ is right-continuous, while $h \mapsto m^-(\beta,h)$ is left-continuous.

Remark 3.30. By Corollary 3.7, $m^+(\beta,h)$ and $m(\beta,h)$ are equal when $h \notin \mathfrak{B}_\beta$. Therefore, considering a sequence $h \downarrow 0$ in $\mathfrak{B}^c_\beta$,

$$m^*(\beta) = \lim_{h\downarrow 0} m(\beta,h) = \lim_{h\downarrow 0} m^+(\beta,h) = m^+(\beta,0) = \langle\sigma_0\rangle^+_{\beta,0}\,.$$

Note also that, by Exercise 3.15,

$$\lim_{\|i\|_1\to\infty} \langle\sigma_0\sigma_i\rangle^+_{\beta,0} = \big(\langle\sigma_0\rangle^+_{\beta,0}\big)^2 = m^*(\beta)^2\,,\qquad \forall\beta\geq 0\,.$$

This observation provides a convenient approach for its explicit computation in $d=2$, which avoids having to work with a nonzero magnetic field. ⋄

Proof. Let $\Lambda_n \Uparrow \mathbb{Z}^d$. By the translation invariance of $\langle\cdot\rangle^+_{\beta,h}$ and by the monotonicity property (3.25),

$$\langle\sigma_0\rangle^+_{\beta,h} = \langle m_{\Lambda_n}\rangle^+_{\beta,h} \leq \langle m_{\Lambda_n}\rangle^+_{\Lambda_n;\beta,h}\,.$$

This gives $\langle\sigma_0\rangle^+_{\beta,h} \leq \liminf_n \langle m_{\Lambda_n}\rangle^+_{\Lambda_n;\beta,h}$. For the other bound, fix $k \geq 1$ and let $i \in \Lambda_n$. On the one hand, if $i + \mathsf{B}(k) \subset \Lambda_n$, (3.25) again gives

$$\langle\sigma_i\rangle^+_{\Lambda_n;\beta,h} \leq \langle\sigma_i\rangle^+_{i+\mathsf{B}(k);\beta,h} = \langle\sigma_0\rangle^+_{\mathsf{B}(k);\beta,h}\,.$$

On the other hand, if $i+\mathsf{B}(k) \not\subset \Lambda_n$, then the box $i+\mathsf{B}(k)$ intersects $\partial^{\text{in}}\Lambda_n$. As a consequence,

$$\begin{aligned}\langle m_{\Lambda_n}\rangle^+_{\Lambda_n;\beta,h} &= \frac{1}{|\Lambda_n|}\sum_{\substack{i\in\Lambda_n:\\ i+\mathsf{B}(k)\subset\Lambda_n}} \langle\sigma_i\rangle^+_{\Lambda_n;\beta,h} + \frac{1}{|\Lambda_n|}\sum_{\substack{i\in\Lambda_n:\\ i+\mathsf{B}(k)\not\subset\Lambda_n}} \langle\sigma_i\rangle^+_{\Lambda_n;\beta,h}\\ &\leq \langle\sigma_0\rangle^+_{\mathsf{B}(k);\beta,h} + 2\frac{|\mathsf{B}(k)||\partial^{\text{in}}\Lambda_n|}{|\Lambda_n|}\,,\end{aligned}$$

since $|\langle\sigma_i\rangle^+_{\Lambda_n;\beta,h}| \leq 1$. (Note that $\langle\sigma_0\rangle^+_{\mathsf{B}(k);\beta,h}$ can be negative; this is the reason for the factor 2 in the last term). This implies that, for all $k \geq 1$, $\limsup_n \langle m_{\Lambda_n}\rangle^+_{\Lambda_n;\beta,h} \leq \langle\sigma_0\rangle^+_{\mathsf{B}(k);\beta,h}$. Since $\lim_{k\to\infty}\langle\sigma_0\rangle^+_{\mathsf{B}(k);\beta,h} = \langle\sigma_0\rangle^+_{\beta,h}$, the desired result follows. The one-sided continuity of $m^+(\beta,h)$ and $m^-(\beta,h)$ will follow from Lemma 3.31 below. □

Lemma 3.31.
1. *For all $\beta \geq 0$, $h \mapsto \langle\sigma_0\rangle^+_{\beta,h}$ is nondecreasing and right-continuous and $h \mapsto \langle\sigma_0\rangle^-_{\beta,h}$ is nondecreasing and left-continuous.*
2. *For all $h \geq 0$, $\beta \mapsto \langle\sigma_0\rangle^+_{\beta,h}$ is nondecreasing and, for all $h \leq 0$, $\beta \mapsto \langle\sigma_0\rangle^-_{\beta,h}$ is nonincreasing.*

Proof of Lemma 3.31: We prove the properties for $\langle\sigma_0\rangle^+_{\beta,h}$ (symmetry then allows us to deduce the corresponding properties for $\langle\sigma_0\rangle^-_{\beta,h}$).

1. Let $\Lambda \Subset \mathbb{Z}^d$. It follows from the FKG inequality that

$$\frac{\partial}{\partial h}\langle\sigma_0\rangle^+_{\Lambda;\beta,h} = \sum_{i\in\Lambda}\bigl(\langle\sigma_0\sigma_i\rangle^+_{\Lambda;\beta,h} - \langle\sigma_0\rangle^+_{\Lambda;\beta,h}\langle\sigma_i\rangle^+_{\Lambda;\beta,h}\bigr) \geq 0\,.$$

So, at fixed Λ, $h \mapsto \langle\sigma_0\rangle^+_{\Lambda;\beta,h}$ is nondecreasing. This monotonicity clearly persists in the thermodynamic limit. Let then $(h_m)_{m\geq 1}$ be a sequence of real numbers such that $h_m \downarrow h$ and $(\Lambda_n)_{n\geq 1}$ be a sequence such that $\Lambda_n \uparrow \mathbb{Z}^d$. Lemma 3.22 implies that the double sequence $(\langle\sigma_0\rangle^+_{\Lambda_n;\beta,h_m})_{m,n\geq 1}$ is nonincreasing and bounded. Consequently, it follows from Lemma B.4 that

$$\begin{aligned}\lim_{m\to\infty}\langle\sigma_0\rangle^+_{\beta,h_m} &= \lim_{m\to\infty}\lim_{n\to\infty}\langle\sigma_0\rangle^+_{\Lambda_n;\beta,h_m}\\ &= \lim_{n\to\infty}\lim_{m\to\infty}\langle\sigma_0\rangle^+_{\Lambda_n;\beta,h_m} = \lim_{n\to\infty}\langle\sigma_0\rangle^+_{\Lambda_n;\beta,h} = \langle\sigma_0\rangle^+_{\beta,h}\,.\end{aligned}$$

The third identity relies on the fact that the finite-volume expectation $\langle\sigma_0\rangle^+_{\Lambda_n;\beta,h}$ is continuous in h.

2. Proceeding as before and using (3.22) with $A = \{0\}$ and $B = \{i,j\}$,

$$\frac{\partial}{\partial\beta}\langle\sigma_0\rangle^+_{\Lambda;\beta,h} = \sum_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda}\bigl(\langle\sigma_0\sigma_i\sigma_j\rangle^+_{\Lambda;\beta,h} - \langle\sigma_0\rangle^+_{\Lambda;\beta,h}\langle\sigma_i\sigma_j\rangle^+_{\Lambda;\beta,h}\bigr) \geq 0\,.$$

This monotonicity also clearly persists in the thermodynamic limit. □

Exercise 3.17. *Let $A \Subset \mathbb{Z}^d$ and $h \geq 0$. Show that $\beta \mapsto \langle\sigma_A\rangle^\varnothing_{\beta,h}$ is left-continuous and $\beta \mapsto \langle\sigma_A\rangle^+_{\beta,h}$ is right-continuous.*

Defining the critical inverse temperature

Since $\langle\sigma_0\rangle^-_{\beta,0} = -\langle\sigma_0\rangle^+_{\beta,0}$ by symmetry, Theorem 3.28 and Remark 3.30 imply that, when $h = 0$, uniqueness is equivalent to $m^*(\beta) = 0$. Since Lemma 3.31 implies that $m^*(\beta) = \langle\sigma_0\rangle^+_{\beta,0}$ is monotone in β, we are led naturally to the following definition.

Definition 3.32. *The **critical inverse temperature** is*

$$\beta_{\mathrm{c}}(d) \stackrel{\text{def}}{=} \inf\bigl\{\beta \geq 0 : m^*(\beta) > 0\bigr\} = \sup\bigl\{\beta \geq 0 : m^*(\beta) = 0\bigr\}\,. \tag{3.27}$$

That is, $\beta_{\mathrm{c}}(d)$ is the unique value of β such that $m^*(\beta) = 0$ if $\beta < \beta_{\mathrm{c}}$, and $m^*(\beta) > 0$ if $\beta > \beta_{\mathrm{c}}$. Of course, one still has to determine whether $\beta_{\mathrm{c}}(d)$ is non-trivial, that is, whether $0 < \beta_{\mathrm{c}}(d) < \infty$.

Remark 3.33. By translation invariance, $\langle\sigma_i\rangle^+_{\beta,0} = \langle\sigma_0\rangle^+_{\beta,0} = m^*(\beta)$ for all $i \in \mathbb{Z}^d$. This implies, using the FKG inequality, that

$$\langle\sigma_0\sigma_i\rangle^+_{\beta,0} \geq \langle\sigma_0\rangle^+_{\beta,0}\langle\sigma_i\rangle^+_{\beta,0} = m^*(\beta)^2\,.$$

In particular,

$$\inf_{i\in\mathbb{Z}^d}\langle\sigma_0\sigma_i\rangle^+_{\beta,0} > 0, \qquad \forall\beta > \beta_{\mathrm{c}}\,. \tag{3.28}$$

Such a behavior is referred to as **long-range order**. The presence of long-range order does not, however, imply that the random variables σ_i display strong correlations at large distances. Indeed, as follows from Exercise 3.15 (see also the more general statement in point 4 of Theorem 6.58), for any β,

$$\lim_{\|i\|_1\to\infty} \langle\sigma_0\sigma_i\rangle^+_{\beta,0} - \langle\sigma_0\rangle^+_{\beta,0}\langle\sigma_i\rangle^+_{\beta,0} = 0\,,$$

so that σ_0 and σ_i are always asymptotically (as $\|i\|_1 \to \infty$) uncorrelated. [3] ⋄

A second characterization of uniqueness

The following theorem provides the promised link between the two notions of first-order phase transition introduced in Definitions 3.8 and 3.24: non-uniqueness occurs at (β, h) if and only if the pressure fails to be differentiable in h at this point. The theorem also provides the extension of the relation (3.8) to values of h at which the pressure is not differentiable. In that case, we can rely on the convexity of the pressure, which we proved in Theorem 3.6, to conclude that its right- and left-derivatives with respect to h are always well defined.

Theorem 3.34. *The following identities hold for all values of $\beta \ge 0$ and $h \in \mathbb{R}$:*

$$\frac{\partial\psi}{\partial h^+}(\beta,h) = m^+(\beta,h), \qquad \frac{\partial\psi}{\partial h^-}(\beta,h) = m^-(\beta,h)\,.$$

In particular, $h \mapsto \psi(\beta,h)$ is differentiable at h if and only if there is a unique Gibbs state at (β,h).

Remark 3.35. Theorem 3.34 shows that the pressure is differentiable with respect to h precisely for those values of β and h at which there is a unique infinite-volume Gibbs state. We will see later (Exercise 6.33) that uniqueness of the infinite-volume Gibbs state also implies differentiability with respect to β. (Actually, although we will not prove it, the pressure of the Ising model on $\mathbb{Z}^d$ is *always* differentiable with respect to β.) ⋄

Proof. Remember that the set $\mathfrak{B}_\beta$ of points of non-differentiability of the pressure is at most countable. Therefore, for each $h \in \mathbb{R}$, it is possible to find a sequence $h_k \downarrow h$ such that $h_k \notin \mathfrak{B}_\beta$ for all $k \ge 1$. It then follows from (3.9) that

$$\frac{\partial\psi}{\partial h^+}(\beta,h) = \lim_{h_k\downarrow h} m(\beta,h_k) = \lim_{h_k\downarrow h} m^+(\beta,h_k) = m^+(\beta,h)\,,$$

since $m^+(\beta,h') = m(\beta,h')$ for all $h' \notin \mathfrak{B}_\beta$ (Corollary 3.7) and $m^+(\beta,h)$ is a right-continuous function of h (Proposition 3.29). Now, by symmetry,

$$\frac{\partial\psi}{\partial h^-}(\beta,h) = -\frac{\partial\psi}{\partial h^+}(\beta,-h) = -m^+(\beta,-h) = m^-(\beta,h)\,.$$

As a consequence, we conclude that

$$\frac{\partial\psi}{\partial h}(\beta,h) \text{ exists} \quad\Leftrightarrow\quad m^+(\beta,h) = m^-(\beta,h)\,.$$

The conclusion follows since, by Proposition 3.29 and Theorem 3.28,

$$m^+(\beta,h) = m^-(\beta,h) \quad\Leftrightarrow\quad \langle\sigma_0\rangle^+_{\beta,h} = \langle\sigma_0\rangle^-_{\beta,h} \quad\Leftrightarrow\quad \text{uniqueness at } (\beta,h). \qquad \square$$

In the following two sections, we prove item 3 of Theorem 3.25 which establishes, at $h = 0$, distinct low- and high-temperature behaviors.

3.7.2 Spontaneous symmetry breaking at low temperatures

In this subsection, we prove that $\beta_c(d) < \infty$, for all $d \ge 2$. In order to do so, it is sufficient to show that, *uniformly in the size of* Λ,

$$\mu^+_{\Lambda;\beta,0}(\sigma_0 = -1) \le \delta(\beta)\,, \tag{3.29}$$

where $\delta(\beta) \downarrow 0$ when $\beta \to \infty$. Indeed, this has the consequence that

$$\begin{aligned}
\langle \sigma_0 \rangle^+_{\Lambda;\beta,0} &= \mu^+_{\Lambda;\beta,0}(\sigma_0 = +1) - \mu^+_{\Lambda;\beta,0}(\sigma_0 = -1) \\
&= 1 - 2\mu^+_{\Lambda;\beta,0}(\sigma_0 = -1) \\
&\ge 1 - 2\delta(\beta)\,.
\end{aligned}$$

Therefore, if one fixes β large enough, so that $1 - 2\delta(\beta) > 0$, and then takes the thermodynamic limit $\Lambda \uparrow \mathbb{Z}^d$, one deduces that

$$m^*(\beta) = \langle \sigma_0 \rangle^+_{\beta,0} > 0\,. \tag{3.30}$$

Using the characterization (3.27), this shows that $\beta_c < \infty$: a first-order phase transition indeed occurs at low temperatures.

The proof of (3.29) uses a key idea originally due to Peierls, today known as *Peierls' argument* and considered a cornerstone in the understanding of phase transitions. It consists in making the following idea rigorous.

When β is large, neighboring spins with different values make a high contribution to the total energy and are thus strongly suppressed. Therefore the contours, *which are the lines that separate regions of + and − spins, should be rare and a typical configuration under $\mu^+_{\Lambda;\beta,0}$ should have the structure of a "sea" of +1 spins with small "islands" of − spins (see Figure 3.10).* ◇

In other words, when β is large, typical configurations under $\mu^+_{\Lambda;\beta,0}$ are small *perturbations* of the ground state η^+, and these perturbations are realized by the contours of the configurations.

We will implement this strategy for the two-dimensional model and will see later how it can be extended to higher dimensions.

Low-temperature representation

Consider the two-dimensional Ising model in $\Lambda \Subset \mathbb{Z}^2$, with zero magnetic field and + boundary condition. We fix some configuration $\omega \in \Omega^+_\Lambda$ and give a geometrical description of ω whose purpose is to account for the above-mentioned fact that a low temperature favors the alignment of nearest-neighbor spins. The starting point is thus to express the Hamiltonian in a way that emphasizes the role played by pairs of opposite spins:

$$\mathscr{H}_{\Lambda;\beta,0} = -\beta \sum_{\{i,j\}\in\mathscr{E}^b_\Lambda} \sigma_i\sigma_j = -\beta|\mathscr{E}^b_\Lambda| + \sum_{\{i,j\}\in\mathscr{E}^b_\Lambda} \beta(1-\sigma_i\sigma_j)\,.$$

The dependence on ω is only in the sum

$$\sum_{\{i,j\}\in\mathscr{E}^b_\Lambda} \beta(1-\sigma_i\sigma_j) = \sum_{\substack{\{i,j\}\in\mathscr{E}^b_\Lambda:\\ \sigma_i\neq\sigma_j}} 2\beta = 2\beta\cdot\#\bigl\{\{i,j\}\in\mathscr{E}^b_\Lambda : \sigma_i\neq\sigma_j\bigr\}\,.$$

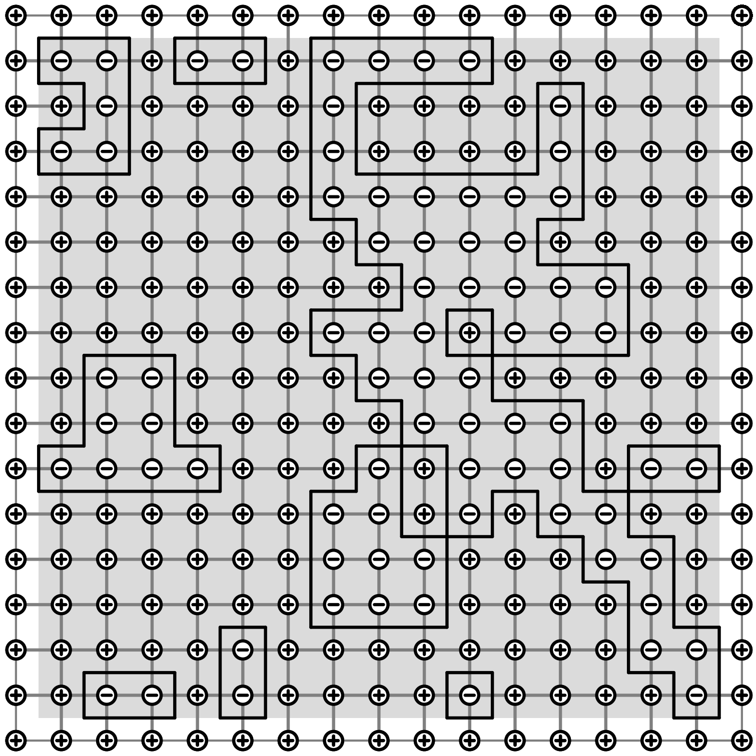


Figure 3.10: A configuration of the two-dimensional Ising model in a finite box Λ with + boundary condition. At low temperature, the lines separating regions of + and − spins are expected to be short and sparse, leading to a positive magnetization in Λ (and thus the validity of (3.29)).

Let us associate to each vertex $i \in \mathbb{Z}^2$ the closed unit square centered at i:

$$\mathscr{S}_i \stackrel{\text{def}}{=} i + [-\tfrac{1}{2}, \tfrac{1}{2}]^2. \tag{3.31}$$

The boundary (in the sense of the standard topology on $\mathbb{R}^2$) of $\mathscr{S}_i$, denoted by $\partial\mathscr{S}_i$, can be considered as being made of four edges connecting nearest-neighbors of the *dual lattice*

$$\mathbb{Z}^2_* = \mathbb{Z}^2 + (\tfrac{1}{2}, \tfrac{1}{2}) \stackrel{\text{def}}{=} \{(i_1 + \tfrac{1}{2}, i_2 + \tfrac{1}{2}) : (i_1, i_2) \in \mathbb{Z}^2\}.$$

Notice that a given edge e of the original lattice intersects exactly one edge $e_\perp$ of the dual lattice. If we associate to a configuration $\omega \in \Omega^+_\Lambda$ the random set

$$\mathscr{M}(\omega) \stackrel{\text{def}}{=} \bigcup_{i \in \Lambda : \sigma_i(\omega) = -1} \mathscr{S}_i,$$

then again $\partial\mathscr{M}(\omega)$ is made of edges of the dual lattice. Moreover, *each edge* $e_\perp = \{i, j\}_\perp \subset \partial\mathscr{M}(\omega)$ *separates two opposite spins:* $\sigma_i(\omega) \neq \sigma_j(\omega)$. One can therefore write

$$\mathscr{H}_{\Lambda;\beta,0}(\omega) = -\beta|\mathscr{E}^{\mathrm{b}}_\Lambda| + 2\beta|\partial\mathscr{M}(\omega)|.$$

(Here, $|\partial\mathscr{M}(\omega)|$ denotes the number of edges contained in $\partial\mathscr{M}(\omega)$ or, equivalently, the total length of $\partial\mathscr{M}(\omega)$.) A configuration ω with its associated set $\partial\mathscr{M}(\omega)$ is represented in Figure 3.10.

We will now decompose $\partial\mathscr{M}(\omega)$ into disjoint components. For that, it is convenient to fix a deformation rule to decide how these components are defined. To this end, we first remark that each dual vertex of $\mathbb{Z}^2_*$ is adjacent to either 0, 2 or 4 edges of $\partial\mathscr{M}(\omega)$ [2]. When this number is 4, we deform $\partial\mathscr{M}(\omega)$ using the following rule:

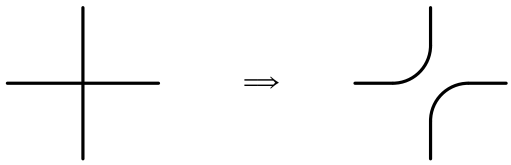


Figure 3.11: The deformation rule.

An application of this rule at all points at which the incidence number is 4 yields a decomposition of $\partial\mathscr{M}(\omega)$ into a set of disjoint closed simple paths on the dual lattice, as in Figure 3.12. In terms of dual edges,

$$\partial\mathscr{M}(\omega) = \gamma_1 \cup \cdots \cup \gamma_n .$$

Each path γ_i is called a **contour of ω**. Let $\Gamma(\omega) \stackrel{\text{def}}{=} \{\gamma_1, \dots, \gamma_n\}$ and define the **length** $|\gamma|$ of a contour $\gamma \in \Gamma(\omega)$ as the number of edges of the dual lattice that it contains. For example, in Figure 3.12, $|\gamma_5| = 14$.

Using the above notations, the energy of a configuration $\omega \in \Omega^+_\Lambda$ can be very simply expressed in terms of its contours:

$$\mathscr{H}_{\Lambda;\beta,0}(\omega) = -\beta|\mathscr{E}^{\mathrm{b}}_\Lambda| + 2\beta \sum_{\gamma\in\Gamma(\omega)} |\gamma| .$$

Consequently, the partition function in Λ with + boundary condition can be written

$$\mathbf{Z}^+_{\Lambda;\beta,0} = e^{\beta|\mathscr{E}^{\mathrm{b}}_\Lambda|} \sum_{\omega\in\Omega^+_\Lambda} \prod_{\gamma\in\Gamma(\omega)} e^{-2\beta|\gamma|} . \tag{3.32}$$

(As usual, the product is defined as equal to 1 when $\Gamma(\omega) = \varnothing$.) Finally, the probability of $\omega \in \Omega^+_\Lambda$ can be expressed in terms of contours as

$$\mu^+_{\Lambda;\beta,0}(\omega) = \frac{e^{-\mathscr{H}_{\Lambda;\beta,0}(\omega)}}{\mathbf{Z}^+_{\Lambda;\beta,0}} = \frac{\prod_{\gamma\in\Gamma(\omega)} e^{-2\beta|\gamma|}}{\sum_\omega \prod_{\gamma\in\Gamma(\omega)} e^{-2\beta|\gamma|}} . \tag{3.33}$$

Remark 3.36. The above probability being a *ratio*, the terms $e^{\beta|\mathscr{E}^{\mathrm{b}}_\Lambda|}$ have canceled out. Therefore, having defined the Hamiltonian without the constant term $\beta|\mathscr{E}^{\mathrm{b}}_\Lambda|$ would have led to the same Gibbs distribution: the energy of a system can always be shifted by a constant without affecting the distribution. ⋄

[2] One way to show that is to consider a dual vertex $x \in \mathbb{Z}^2_*$ together with the four surrounding points of $\mathbb{Z}^2$, which we denote (in clockwise order) by i, j, k, l. Since $(\omega_i\omega_j)(\omega_j\omega_k)(\omega_k\omega_l)(\omega_l\omega_i) = \omega_i^2\omega_j^2\omega_k^2\omega_l^2 = 1$, the number of products equal to -1 in the leftmost expression is even. But such a product is equal to -1 precisely when the edge of a contour separates the corresponding spins.

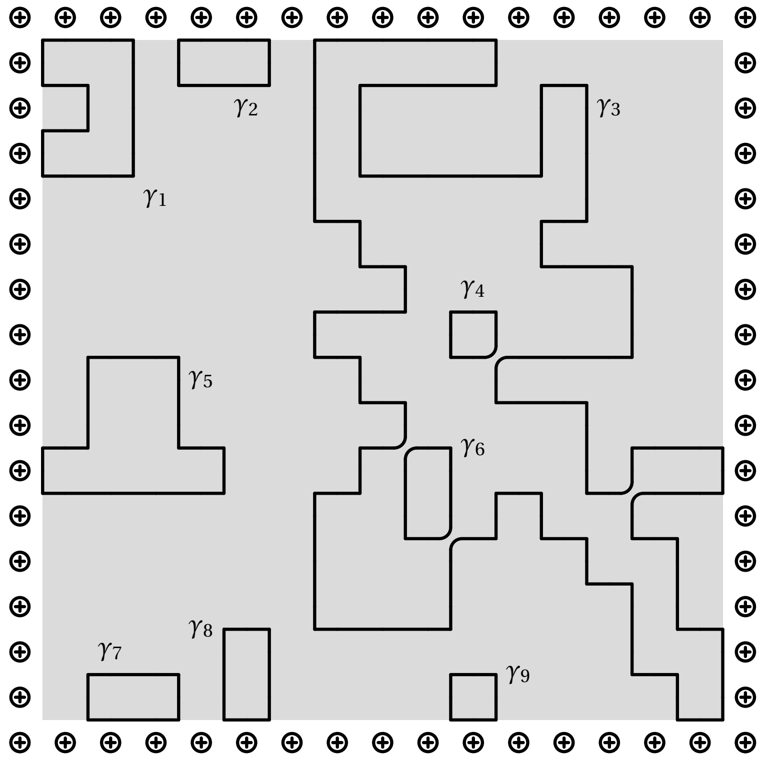


Figure 3.12: The contours (paths on the dual lattice) associated to the configuration of Figure 3.10. Together with the value of the spins on the boundary (+1 in the present case), the original configuration ω can be reconstructed in a unique manner.

Peierls' argument

We consider the box $\mathsf{B}(n) = \{-n,\dots,n\}^2$. To study $\mu^+_{\mathsf{B}(n);\beta,0}(\sigma_0 = -1)$, we first observe that *any configuration* $\omega \in \Omega^+_{\mathsf{B}(n)}$ *such that* $\sigma_0(\omega) = -1$ *must possess at least one (actually, an odd number of) contours surrounding the origin.*

To make this statement precise, notice that each contour $\gamma \in \Gamma(\omega)$ is a bounded simple closed curve in $\mathbb{R}^2$ and therefore splits the plane into two regions, exactly one of which is bounded, called the **interior** of γ and denoted $\mathrm{Int}(\gamma)$. We can thus write

$$\mu^+_{\mathsf{B}(n);\beta,0}(\sigma_0 = -1) \le \mu^+_{\mathsf{B}(n);\beta,0}\big(\exists \gamma_* \in \Gamma : \mathrm{Int}(\gamma_*) \ni 0\big) \le \sum_{\gamma_* : \mathrm{Int}(\gamma_*) \ni 0} \mu^+_{\mathsf{B}(n);\beta,0}(\Gamma \ni \gamma_*)\,.$$

Lemma 3.37. *For all* $\beta > 0$ *and any contour* γ_*,

$$\mu^+_{\mathsf{B}(n);\beta,0}(\Gamma \ni \gamma_*) \le e^{-2\beta|\gamma_*|}\,. \tag{3.34}$$

The bound (3.34) shows that the probability that a given contour appears in a configuration becomes small when β is large or when the contour is long. Later, we will refer to such a fact by saying that the ground state η^+ is *stable*.

Proof of Lemma 3.37. Using (3.33),

$$\mu^+_{\mathsf{B}(n);\beta,0}(\Gamma \ni \gamma_*) = \sum_{\omega:\Gamma(\omega)\ni\gamma_*} \mu^+_{\mathsf{B}(n);\beta,0}(\omega)$$
$$= e^{-2\beta|\gamma_*|} \frac{\sum_{\omega:\Gamma(\omega)\ni\gamma_*} \prod_{\gamma\in\Gamma(\omega)\setminus\{\gamma_*\}} e^{-2\beta|\gamma|}}{\sum_\omega \prod_{\gamma\in\Gamma(\omega)} e^{-2\beta|\gamma|}}. \tag{3.35}$$

We will show that the ratio in (3.35) is bounded above by 1, by proving that the sum in the numerator is the same as the one in the denominator, but with an additional constraint. To each configuration ω with $\Gamma(\omega) \ni \gamma_*$ appearing in the sum of the numerator, we associate the configuration $\mathscr{E}_{\gamma_*}(\omega)$ obtained from ω by "removing γ_*". This can be done by simply flipping all spins in the interior of γ_*:

$$(\mathscr{E}_{\gamma_*}(\omega))_i \stackrel{\text{def}}{=} \begin{cases} -\omega_i & \text{if } i \in \mathrm{Int}(\gamma_*), \\ \omega_i & \text{otherwise.} \end{cases} \tag{3.36}$$

It is important to realize that $\mathscr{E}_{\gamma_*}(\omega)$ is the configuration whose set of contours is exactly $\Gamma(\omega) \setminus \{\gamma_*\}$. For instance, even if $\mathrm{Int}(\gamma_*)$ contains other contours (as γ_3 in Figure 3.12, which contains γ_4 and γ_6 in its interior), these continue to exist after flipping the spins. Let then $\mathfrak{C}(\gamma_*)$ be the set of configurations that can be obtained by removing γ_* from a configuration containing γ_*. We have

$$\sum_{\omega:\Gamma(\omega)\ni\gamma_*} \prod_{\gamma\in\Gamma(\omega)\setminus\{\gamma_*\}} e^{-2\beta|\gamma|} = \sum_{\omega'\in\mathfrak{C}(\gamma_*)} \prod_{\gamma'\in\Gamma(\omega')} e^{-2\beta|\gamma'|}.$$

But since the sum over $\omega' \in \mathfrak{C}(\gamma_*)$ is less than the sum over all $\omega' \in \Omega^+_{\mathsf{B}(n)}$, this shows that the ratio in (3.35) is indeed bounded above by 1. □

Each of the sums in the ratio in (3.35) *is typically exponentially large or small in* $|\mathsf{B}(n)|$. *We have proved that the ratio is nevertheless bounded above by* 1 *by flipping the spins of the configuration inside the contour* γ_**, an operation that relied crucially on the symmetry of the model under a global spin flip.* ⋄

Using (3.34), we bound the sum over all contours that surround the origin, by grouping them according to their lengths. Since the smallest contour surrounding the origin is made of 4 dual edges,

$$\mu^+_{\mathsf{B}(n);\beta,0}(\sigma_0 = -1) \le \sum_{\gamma_*:\mathrm{Int}(\gamma_*)\ni 0} e^{-2\beta|\gamma_*|} \tag{3.37}$$
$$= \sum_{k\ge4} \sum_{\substack{\gamma_*:\mathrm{Int}(\gamma_*)\ni 0\\ |\gamma_*|=k}} e^{-2\beta|\gamma_*|}$$
$$= \sum_{k\ge4} e^{-2\beta k} \#\{\gamma_* : \mathrm{Int}(\gamma_*) \ni 0, |\gamma_*| = k\}. \tag{3.38}$$

A contour of length k surrounding the origin necessarily contains a vertex of the set $\{(u - \frac{1}{2}, \frac{1}{2}) : u = 1, \dots, [k/2]\}$. But the total number of contours of length k starting from a given vertex is at most $4 \cdot 3^{k-1}$. Indeed, there are 4 available directions for the first segment, then at most 3 for each of the remaining $k-1$ segments (since the contour does not use twice the same edge). Therefore,

$$\#\{\gamma_* : \mathrm{Int}(\gamma_*) \ni 0, |\gamma_*| = k\} \le \tfrac{k}{2} \cdot 4 \cdot 3^{k-1}. \tag{3.39}$$

Gathering these estimates,

$$\mu^+_{\mathsf{B}(n);\beta,0}(\sigma_0 = -1) \le \tfrac{2}{3} \sum_{k\ge 4} k 3^k e^{-2\beta k} \stackrel{\text{def}}{=} \delta(\beta). \tag{3.40}$$

If β is large enough (so that $3e^{-2\beta} < 1$), then the series in (3.40) converges. Moreover, $\delta(\beta) \downarrow 0$ as $\beta \to \infty$. This proves (3.29), which concludes the proof that $\beta_c(2) < \infty$.

Before turning to the case $d \ge 3$, let us see what additional information about the low-temperature behavior of the two-dimensional Ising model can be extracted using the approach discussed above. The next exercise shows that, at sufficiently low temperatures, typical configurations in $\mathsf{B}(n)$, for the model with + boundary condition, consist of a "sea" of + spins with small islands of − spins (the latter possibly containing "lakes" of + spins, etc.). Namely, the largest contour in $\mathsf{B}(n)$ has a length of order $\log n$.

Exercise 3.18. *Consider the two-dimensional Ising model.*

1. *Show that there exists $\beta_0 < \infty$ such that the following holds for all $\beta > \beta_0$. For any $c > 0$, there exists $K_0(c) < \infty$ such that, for all $K > K_0(c)$ and all n,*

$$\mu^+_{\mathsf{B}(n);\beta,0}\big(\exists \gamma \in \Gamma \text{ with } |\gamma| \ge K \log n\big) \le n^{-c}.$$

2. *Show that, for all $\beta \ge 0$ and all $c > 0$, there exist $K_1(\beta, c) > 0$ and $n_0(c) < \infty$ such that, for all $K < K_1(\beta, c)$ and all $n \ge n_0(c)$,*

$$\mu^+_{\mathsf{B}(n);\beta,0}\big(\exists \gamma \in \Gamma \text{ with } |\gamma| \ge K \log n\big) \ge 1 - e^{-n^{2-c}}.$$

Introducing a positive magnetic field h should only make the appearance of contours less likely, so that it is natural to expect that the claims of the previous exercise still hold in that case.

Exercise 3.19. *Extend the claims of Exercise 3.18 to the case $h > 0$.* Hint: *For the first claim, observe that the existence of a long contour implies the existence of a long path of − spins, which is a decreasing event; then use the FKG inequality.*

Extension to larger dimensions. It remains to show that a phase transition also occurs in the Ising model in dimensions $d \ge 3$. Adapting Peierls' argument to higher dimensions is possible, but the counting in (3.39) becomes a little trickier.

Exercise 3.20. *Show that Peierls' estimate can be extended to $\mathbb{Z}^d$, $d \ge 3$. The combinatorial estimate on the sum of contours can be done using Lemma 3.38 below.*

Nevertheless, we will analyze the model in $d \ge 3$ by following an alternative approach: using the natural embedding of $\mathbb{Z}^d$ into $\mathbb{Z}^{d+1}$ and the GKS inequalities, we will prove that $\beta_c(d)$ is nonincreasing in d.

The idea is elementary: one can build the Ising model on $\mathbb{Z}^{d+1}$ by considering a stack of Ising models on $\mathbb{Z}^d$ and adding interactions between neighboring spins *living in successive layers. Then, the GKS inequalities tell us that adding these interactions does not decrease the magnetization and, thus, does not increase the inverse critical temperature.* ⋄

To simplify notation, we treat explicitly only the case $d = 3$; the extension to higher dimensions is straightforward. We will see $\mathbb{Z}^2$ as embedded in $\mathbb{Z}^3$. Therefore, we will temporarily use the following notations:

$$\mathsf{B}^3(n) \stackrel{\text{def}}{=} \{-n,\dots,n\}^3, \qquad \mathsf{B}^2(n) \stackrel{\text{def}}{=} \{-n,\dots,n\}^2.$$

We claim that

$$\langle\sigma_0\rangle^+_{\mathsf{B}^3(n);\beta,0} \geq \langle\sigma_0\rangle^+_{\mathsf{B}^2(n);\beta,0}.$$

Namely, consider the set of edges $\{i,j\}$ connecting two nearest-neighbor vertices $i = (i_1,i_2,i_3)$ and $j = (j_1,j_2,j_3)$ such that $i_3 = 0$ and $|j_3| = 1$. The two spins living at the endpoints of such an edge contribute to the total energy by an amount $-\beta\sigma_i\sigma_j \equiv -J_{ij}\sigma_i\sigma_j$ (remember the Hamiltonian written as in (3.20)). Thanks to the GKS inequalities,

$$\frac{\partial}{\partial J_{ij}}\langle\sigma_0\rangle^+_{\mathsf{B}^3(n);\beta,0} = \langle\sigma_0\sigma_i\sigma_j\rangle^+_{\mathsf{B}^3(n);\beta,0} - \langle\sigma_0\rangle^+_{\mathsf{B}^3(n);\beta,0}\langle\sigma_i\sigma_j\rangle^+_{\mathsf{B}^3(n);\beta,0} \geq 0.$$

We can therefore consider those edges, one after the other, and for each of them gradually decrease the interaction from its initial value $J_{ij} = \beta$ down to $J_{ij} = 0$. Denoting by $\mu^{+,0}_{\mathsf{B}^3(n);\beta,0}$ the Gibbs distribution obtained after all those coupling constants J_{ij} have been brought down to zero, we obtain

$$\langle\sigma_0\rangle^+_{\mathsf{B}^3(n);\beta,0} \geq \langle\sigma_0\rangle^{+,0}_{\mathsf{B}^3(n);\beta,0}.$$

Observe that the spins contained in the layer $\{j_3 = 0\}$ interact now as if they were in a two-dimensional system, and so $\langle\sigma_0\rangle^{+,0}_{\mathsf{B}^3(n);\beta,0} = \langle\sigma_0\rangle^+_{\mathsf{B}^2(n);\beta,0}$. We therefore get

$$\lim_{n\to\infty}\langle\sigma_0\rangle^+_{\mathsf{B}^3(n);\beta,0} \geq \lim_{n\to\infty}\langle\sigma_0\rangle^+_{\mathsf{B}^2(n);\beta,0}.$$

Combined with (3.27), this inequality shows, in particular, that $\beta_c(3) \leq \beta_c(2)$. The existence of a first-order phase transition at low temperatures for the Ising model on $\mathbb{Z}^3$ thus follows from the already proven fact that $\beta_c(2) < \infty$.

Improved bound. It is known that the inverse critical temperature of the two-dimensional Ising model equals

$$\beta_c(2) = \tfrac{1}{2}\operatorname{arcsinh}(1) \cong 0.441.$$

Obviously, not much care was taken, in our application of Peierls' argument, to optimize the resulting upper bound on $\beta_c(2)$. The following exercise shows how a slightly more careful application of the same ideas can lead to a rather decent upper bound (with a relative error of order 10%, compared to the exact value).

Revised version, August 22 2017
To be published by Cambridge University Press (2017)

www.unige.ch/math/folks/velenik/smbook

Exercise 3.21. 1. *Check, using* (3.40)*, that* $\beta_c(2) < 0.88$.

2. *The aim of this exercise is to improve this estimate to* $\beta_c(2) < 0.493$. *This will be done by showing that*
$$\beta_c(2) \le \tfrac{1}{2}\log\mu\,,$$
where μ *is the connectivity constant of* $\mathbb{Z}^2$*, defined by*
$$\mu \stackrel{\text{def}}{=} \lim_{n\to\infty}\frac{1}{n}\log C_n\,,$$
where C_n *is the number of nearest-neighbor paths of length* n *starting at the origin and visiting each of its vertices at most once. It is known that* $2.62 < \mu < 2.68$. Hint: *Proceed similarly as in* (3.37) *and show that the ratio*
$$\frac{\mu^+_{\mathrm{B}(n);\beta,0}\big(\sigma_i = -1\,\forall i\in\mathrm{B}(R)\big)}{\mu^-_{\mathrm{B}(n);\beta,0}\big(\sigma_i = -1\,\forall i\in\mathrm{B}(R)\big)} < 1\,,$$
uniformly in n*, provided that* $\beta > \frac{1}{2}\log\mu$ *and that* R *is large enough.*

3.7.3 Uniqueness at high temperature

There exist several distinct methods to prove that there is a unique Gibbs state when the spins are weakly dependent, that is, at high temperatures. Two general approaches will be presented in Section 6.5. Here, we will rely on a graphical representation, which is well adapted to a description of high-temperatures correlations.

High-temperature representation. This representation relies on the following elementary identity. Since $\sigma_i\sigma_j$ only takes the two values ±1,
$$e^{\beta\sigma_i\sigma_j} = \cosh(\beta) + \sigma_i\sigma_j\sinh(\beta) = \cosh(\beta)\big(1+\tanh(\beta)\sigma_i\sigma_j\big)\,. \tag{3.41}$$
Identity (3.41) can be used to rewrite the Boltzmann weight. For all $\Lambda \Subset \mathbb{Z}^d$ and $\omega\in\Omega^+_\Lambda$,
$$e^{-\mathscr{H}_{\Lambda;\beta,0}(\omega)} = \prod_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda} e^{\beta\sigma_i(\omega)\sigma_j(\omega)} = \cosh(\beta)^{|\mathscr{E}^{\mathrm{b}}_\Lambda|}\prod_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda}\big(1+\tanh(\beta)\omega_i\omega_j\big)\,, \tag{3.42}$$
where $\mathscr{E}^{\mathrm{b}}_\Lambda$ was defined in (3.2). We will now expand the product over the edges.

Exercise 3.22. *Show that, for any nonempty finite set* $\mathscr{E}$,
$$\prod_{e\in\mathscr{E}}(1+f(e)) = \sum_{E\subset\mathscr{E}}\prod_{e\in E} f(e)\,. \tag{3.43}$$
(As usual, the product in the right-hand side is 1 *if* $E=\varnothing$*.)*

Using (3.43) in (3.42) and changing the order of summation, we get
$$\mathbf{Z}^+_{\Lambda;\beta,0} = \cosh(\beta)^{|\mathscr{E}^{\mathrm{b}}_\Lambda|}\sum_{E\subset\mathscr{E}^{\mathrm{b}}_\Lambda}\tanh(\beta)^{|E|}\sum_{\omega\in\Omega^+_\Lambda}\underbrace{\prod_{\{i,j\}\in E}\omega_i\omega_j}_{=\prod_{i\in\Lambda}\omega_i^{I(i,E)}}\,,$$

where $I(i,E)$ is the incidence number: $I(i,E) \stackrel{\text{def}}{=} \#\{j \in \mathbb{Z}^d : \{i,j\} \in E\}$. Now, the summation over $\omega \in \Omega_\Lambda^+$ can be made separately for each vertex $i \in \Lambda$:

$$\sum_{\omega_i=\pm 1} \omega_i^{I(i,E)} = \begin{cases} 2 & \text{if } I(i,E) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.44}$$

We conclude that

$$\mathbf{Z}^+_{\Lambda;\beta,0} = 2^{|\Lambda|} \cosh(\beta)^{|\mathscr{E}^{\mathrm{b}}_\Lambda|} \sum_{E \in \mathfrak{E}^{+;\mathrm{even}}_\Lambda} \tanh(\beta)^{|E|}, \tag{3.45}$$

where

$$\mathfrak{E}^{+;\mathrm{even}}_\Lambda \stackrel{\text{def}}{=} \{E \subset \mathscr{E}^{\mathrm{b}}_\Lambda : I(i,E) \text{ is even for all } i \in \Lambda\}.$$

When convenient, we will identify such sets of edges with the graph they induce [3].

The expression (3.45) is called the *high-temperature representation* of the partition function. Proceeding in the same manner, we see that $\langle\sigma_0\rangle^+_{\Lambda;\beta,0}$ can be written

$$\begin{aligned} \langle\sigma_0\rangle^+_{\Lambda;\beta,0} &= \big(\mathbf{Z}^+_{\Lambda;\beta,0}\big)^{-1} 2^{|\Lambda|} \cosh(\beta)^{|\mathscr{E}^{\mathrm{b}}_\Lambda|} \sum_{E \in \mathfrak{E}^{+;0}_\Lambda} \tanh(\beta)^{|E|} \\ &= \frac{\sum_{E \in \mathfrak{E}^{+;0}_\Lambda} \tanh(\beta)^{|E|}}{\sum_{E \in \mathfrak{E}^{+;\mathrm{even}}_\Lambda} \tanh(\beta)^{|E|}}, \end{aligned} \tag{3.46}$$

where

$$\mathfrak{E}^{+;0}_\Lambda \stackrel{\text{def}}{=} \{E \subset \mathscr{E}^{\mathrm{b}}_\Lambda : I(i,E) \text{ is even for all } i \in \Lambda \setminus \{0\}, \text{ but } I(0,E) \text{ is odd}\}.$$

Given $E \subset \mathscr{E}^{\mathrm{b}}_\Lambda$, we denote by $\Delta(E)$ the set of all edges of $\mathscr{E}^{\mathrm{b}}_\Lambda$ sharing no endpoint with an edge of E. Any collection of edges $E \in \mathfrak{E}^{+;0}_\Lambda$ can then be decomposed as $E = E_0 \cup E'$, with $E_0 \neq \varnothing$ the connected component of E containing 0, and $E' \in \mathfrak{E}^{+;\mathrm{even}}_\Lambda$ satisfying $E' \subset \Delta(E_0)$. Therefore,

$$\langle\sigma_0\rangle^+_{\Lambda;\beta,0} = \sum_{\substack{E_0 \in \mathfrak{E}^{+;0}_\Lambda \\ \text{connected}, E_0 \ni 0}} \tanh(\beta)^{|E_0|} \frac{\sum_{E' \in \mathfrak{E}^{+;\mathrm{even}}_\Lambda : E' \subset \Delta(E_0)} \tanh(\beta)^{|E'|}}{\sum_{E \in \mathfrak{E}^{+;\mathrm{even}}_\Lambda} \tanh(\beta)^{|E|}}. \tag{3.47}$$

Proof that $\beta_c(d) > 0$, for all d. Bounding the ratio in (3.47) by 1,

$$\langle\sigma_0\rangle^+_{\mathrm{B}(n);\beta,0} \leq \sum_{\substack{E_0 \in \mathfrak{E}^{+;0}_{\mathrm{B}(n)} \\ \text{connected}, E_0 \ni 0}} \tanh(\beta)^{|E_0|}. \tag{3.48}$$

The sum can be bounded using the following lemma.

Lemma 3.38. *Let G be a connected graph with N edges. Starting from an arbitrary vertex of G, there exists a path in G crossing each edge of G exactly twice.*

[3] The graph induced by a set E of edges is the graph having E as edges and the endpoints of the edges in E as vertices.

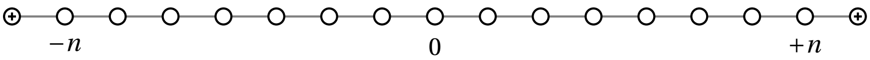
Proof. The proof proceeds by induction on N, observing that an arbitrary connected graph can always be built one edge at a time in such a way that all intermediate graphs are also connected. When $N = 1$, the result is trivial. Suppose that the result holds when $N = k$ and let $\pi = (\pi(1), \dots, \pi(2k))$ be one of the corresponding paths. We add a new edge to the graph, keeping it connected; this implies that at least one endpoint v of this edge belongs to the original graph. The desired path is obtained by following π until the first visit at v, then crossing the new edge once in each direction, and finally following the path π to its end. □

Using this lemma, we see that the number of graphs E_0 with ℓ edges contributing to (3.48) is bounded above by the number of paths of length 2ℓ starting from 0. The latter is certainly smaller than $(2d)^{2\ell}$ since each new edge can be taken in at most $2d$ different directions. On the other hand, E_0 connects necessarily 0 to $\mathsf{B}(n)^c$: Indeed, $\sum_{i\in\mathbb{Z}^d} I(i,E_0) = 2|E_0|$ is even; since $I(0,E_0)$ is odd, there must be at least one vertex $i \neq 0$ with $I(i,E_0)$ odd; however, such a vertex cannot belong to $\mathsf{B}(n)$, since $I(i,E_0)$ is even for all $i \in \mathsf{B}(n) \setminus \{0\}$. We conclude that $|E_0| \geq n$, which yields, since $\tanh(\beta) \leq \beta$,

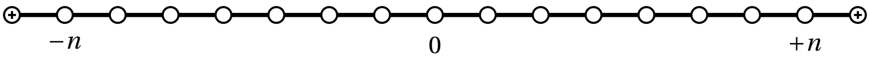
$$\langle\sigma_0\rangle^+_{\mathsf{B}(n);\beta,0} \leq \sum_{\ell\geq n} (4d^2\beta)^\ell \leq e^{-cn}, \tag{3.49}$$

with $c = c(\beta,d) > 0$, for all $\beta < 1/(4d^2)$. In particular, $\langle\sigma_0\rangle^+_{\beta,0} = 0$ for all $\beta < 1/(4d^2)$, which implies that $\beta_c(d) > 0$, that is, uniqueness at high temperatures, by Theorem 3.28 and the characterization (3.27).

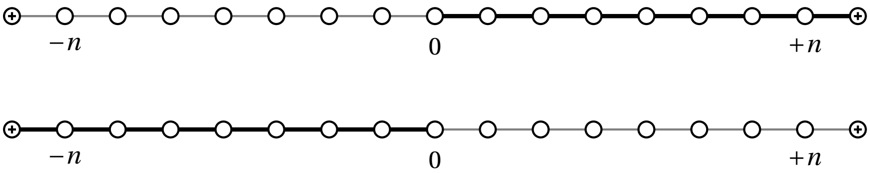
Proof that $\beta_c(1) = +\infty$. Consider the Ising model in a one-dimensional box $\mathsf{B}(n)$ with + boundary condition:



Due to the structure of $\mathbb{Z}$, there are only few subgraphs of $E \subset \mathscr{E}^b_{\mathsf{B}(n)}$ appearing in the ratio (3.46) and they are particularly simple. We first consider the denominator. Since the subgraphs appearing in the sum must be such that the incidence number of each $i \in \mathsf{B}(n)$ is either 0 or 2, $\mathfrak{E}^{+;\text{even}}_{\mathsf{B}(n)}$ can contain only two graphs: the graph whose set of edges is $E = \varnothing$, as in the previous figure, and the one for which $E = \mathscr{E}^b_{\mathsf{B}(n)}$:



On the other hand, $\mathfrak{E}^{+;0}_{\mathsf{B}(n)}$ also reduces to two graphs, one composed of all edges with two nonnegative endpoints, and one composed of all edges with two nonpositive endpoints:



Revised version, August 22 2017
To be published by Cambridge University Press (2017)

www.unige.ch/math/folks/velenik/smbook

Consequently, (3.46) becomes

$$\langle\sigma_0\rangle^+_{\mathsf{B}(n);\beta,0} = \frac{2\tanh(\beta)^{n+1}}{1+\tanh(\beta)^{2(n+1)}},$$

which indeed tends to 0 when $n\to\infty$, for all $\beta<\infty$.

Exercise 3.23. *Derive representations similar to* (3.47) *for* $\langle\sigma_i\sigma_j\rangle^+_{\Lambda;\beta,0}$, $\mathbf{Z}^\varnothing_{\Lambda;\beta,0}$ *and* $\langle\sigma_i\sigma_j\rangle^\varnothing_{\Lambda;\beta,0}$.

The next exercise shows that the 2-point function decays exponentially when β is small enough.

Exercise 3.24. *Using Exercise 3.23, prove that, for all β sufficiently small, there exists $c=c(\beta)>0$ such that $\langle\sigma_i\sigma_j\rangle_{\beta,0}\le e^{-c\|j-i\|_1}$, for all $i,j\in\mathbb{Z}^d$, where $\langle\cdot\rangle_{\beta,0}$ is the unique Gibbs state.*

Note that, as shown in the next exercise, the decay of the 2-point function can never be faster than exponential (when $\beta\neq 0$):

Exercise 3.25. *Using the GKS inequalities, prove that, in any dimension $d\ge 1$ and at any $\beta\ge 0$,*

$$\langle\sigma_i\sigma_j\rangle^+_{\beta,0}\ge\langle\sigma_i\sigma_j\rangle^\varnothing_{\beta,0}\ge\langle\sigma_0\sigma_{\|j-i\|_1}\rangle^{d=1}_{\Lambda_{i,j};\beta,0},$$

where the expectation in the right-hand side is with respect to the Gibbs distribution with free boundary condition in the box $\Lambda_{ij}=\{0,\dots,\|j-i\|_1\}\subset\mathbb{Z}$. Using Exercise 3.23, show that the 2-point function in the right-hand side is equal to $(\tanh\beta)^{\|j-i\|_1}$.

Remark 3.39. It is actually possible to prove that the exponential decay of $\langle\sigma_i\sigma_j\rangle_{\beta,0}$ and the exponential relaxation of $\langle\sigma_0\rangle^+_{\mathsf{B}(n);\beta,0}$ toward $\langle\sigma_0\rangle^+_{\beta,0}$ hold true for all $\beta<\beta_c(d)$. [3] ⋄

Exercise 3.26. *Use the high-temperature representation as an alternative way of computing the pressure of the one-dimensional Ising model with $h=0$. Compare the expressions for $\psi^+_{\mathsf{B}(n)}$, $\psi^\varnothing_{\mathsf{B}(n)}$ and $\psi^{\mathrm{per}}_{\mathsf{B}(n)}$.*

3.7.4 Uniqueness in nonzero magnetic field

We are now left with the proof of item 1 of Theorem 3.25, which states that, when $h\neq 0$, the Gibbs state associated to (β,h) is always unique, regardless of the value of β. The proof will take us on a detour, using results from complex analysis, and will allow us to establish a very strong property of the pressure of the Ising model.

We will study the existence and properties of the pressure when h takes values in the complex domains

$$H^+\stackrel{\text{def}}{=}\{z\in\mathbb{C}:\mathfrak{Re}\,z>0\},$$
$$H^-\stackrel{\text{def}}{=}\{z\in\mathbb{C}:\mathfrak{Re}\,z<0\}.$$

Since the inverse temperature $\beta>0$ will play no particular role in this section, we will omit it from the notations at some places. For example, we will write $\psi(h)$ rather than $\psi(\beta,h)$.

Theorem 3.40. *Let $\beta > 0$. As a function of the magnetic field h, the pressure of the Ising model in the thermodynamic limit, $\psi = \psi(h)$, can be extended from $\{h \in \mathbb{R} : h > 0\}$ (resp. $\{h \in \mathbb{R} : h < 0\}$) to an analytic function on the whole domain H^+ (resp. H^-). On H^+ and H^-, ψ can be computed using the thermodynamic limit with free boundary condition.*

This result of course implies that the complex derivative of ψ with respect to h exists on H^+ and H^-. Therefore, the real partial derivative $\frac{\partial \psi}{\partial h}$ exists at each real $h \neq 0$. By Theorem 3.34, this implies uniqueness of the Gibbs state for all $h \neq 0$, thus completing the proof of Theorem 3.25.

Remark 3.41. The GHS inequality, which is not discussed in this book, allows to give an alternative proof of the differentiability of the pressure when $h \neq 0$, avoiding complex analysis. Namely, the GHS inequality can be used to show that the magnetization $h \mapsto \langle \sigma_0 \rangle^+_{\beta,h}$ is concave, and hence continuous, on $\mathbb{R}_{\geq 0}$. This implies that its antiderivative (which is equal to ψ up to a constant) exists and is differentiable on $(0, +\infty)$. Of course, combined with Theorem 3.34, Theorem 3.40 implies the much stronger statement that $h \mapsto \langle \sigma_0 \rangle^+_{\beta,h}$ is real analytic on $\{h < 0\}$ and $\{h > 0\}$. ⋄

We have seen that, for real parameters, the thermodynamic limit of the pressure can be computed using an arbitrary boundary condition. When the magnetic field is complex, the boundary condition becomes a nuisance. It turns out that the free boundary condition is particularly convenient. We will therefore work in finite volumes $\Lambda \Subset \mathbb{Z}^d$ and study

$$\psi^{\varnothing}_{\Lambda}(h) = \frac{1}{|\Lambda|} \log \mathbf{Z}^{\varnothing}_{\Lambda;\beta,h} .$$

The existence and analyticity properties of the pressure are established by taking the thermodynamic limit $\Lambda \Uparrow \mathbb{Z}^d$ for this choice of boundary condition. The analytic function obtained is then the analytic continuation of the pressure to complex values of the field [4].

On the one hand when the magnetic field is real, $\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}$ being a finite linear combination of powers of $e^{\pm h}$, is real-analytic in h. Moreover, since

$$\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h} > 0 \quad \text{for all } h \in \mathbb{R}, \tag{3.50}$$

the pressure $\psi^{\varnothing}_{\Lambda}(\cdot)$ is also real-analytic in h. It is not true, however, that this real analyticity always holds in the thermodynamic limit $\Lambda \Uparrow \mathbb{Z}^d$. Indeed, we have seen (using Peierls' argument) that, at low temperature, the pressure is not even differentiable at $h = 0$.

On the other hand, since the Boltzmann weights are complex numbers when $h \in \mathbb{C}$, the partition function $\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}$ can very well vanish, leading to a problem even for the definition of the finite-volume pressure.

Fortunately, the celebrated *Lee–Yang Circle Theorem*, Theorem 3.43, will show that the partition function satisfies a remarkable property, analogous to (3.50), in suitable domains of the complex plane. This will allow us to control the analyticity of the pressure in the thermodynamic limit, as explained in the following result.

[4] We remind the reader of the following fact: if two functions analytic on a domain D coincide on a set $A \subset D$ which has an accumulation point in D, then these two functions are equal on D. Therefore, if it were possible to obtain another pressure $\tilde{\psi}$ using a different boundary condition, analytic on H^+ and H^-, then, since this pressure coincides with the one obtained with free boundary conditions on the real axis, it must coincide with it on H^+ and H^-.

Theorem 3.42 (Lee–Yang)**.** *Let $\beta \ge 0$. Let $D \subset \mathbb{C}$ be open, simply connected and such that $D \cap \mathbb{R}$ is an interval of $\mathbb{R}$. Assume that, for every finite volume $\Lambda \Subset \mathbb{Z}^d$,*

$$\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h} \neq 0 \quad \forall h \in D. \tag{3.51}$$

Then, the pressure $h \mapsto \psi(h)$ admits an analytic continuation to D.

We know from the analysis done in the previous sections that the pressure is not differentiable at $h = 0$ when $\beta > \beta_c(d)$. When this happens, the previous theorem implies that there must exist a sequence $(h_k) \in \mathbb{C}$, tending to 0 and a sequence $\Lambda_k \Uparrow \mathbb{Z}^d$ such that $\mathbf{Z}^{\varnothing}_{\Lambda_k;\beta,h_k} = 0$ for all k. Therefore, even though the partition functions never vanish as long as h is real, complex zeroes approach the point $h = 0$ in the thermodynamic limit. In this sense, although values of the magnetic field with a nonzero imaginary part may be experimentally meaningless [4], the way the partition function behaves for such complex values of the magnetic field turns out to have profound physical consequences.

Proof of Theorem 3.42: (The precise statements of the few classical results of complex analysis needed in the proof below can be found in Appendix B.3.)

Let $\Lambda_n \Uparrow \mathbb{Z}^d$. Using (3.51), Theorem B.23 guarantees that one can find a function $h \mapsto \log \mathbf{Z}^{\varnothing}_{\Lambda_n;\beta,h}$ analytic on D and coinciding with the quantity studied in the rest of this chapter when $h \in D \cap \mathbb{R}$ (see Remark B.24 for the existence of a branch of the logarithm with this property). One can then define

$$g_n(h) \stackrel{\text{def}}{=} \exp\bigl(|\Lambda_n|^{-1} \log \mathbf{Z}^{\varnothing}_{\Lambda_n;\beta,h}\bigr),$$

which is also analytic on D. Now, when $h \in D \cap \mathbb{R}$, $g_n(h)$ coincides with $e^{\psi^{\varnothing}_{\Lambda_n}(h)}$, and Theorem 3.6 thus guarantees that, for such values of h, $g_n(h) \to g(h) \stackrel{\text{def}}{=} e^{\psi(h)}$ as $n \to \infty$, where ψ is the pressure of the Ising model in infinite volume.

The next observation is that the sequence (g_n) is locally uniformly bounded on D, since

$$\begin{aligned} |\mathbf{Z}^{\varnothing}_{\Lambda_n;\beta,h}| &\le \sum_{\omega \in \Omega_{\Lambda_n}} \bigl|\exp\bigl(-\mathscr{H}^{\varnothing}_{\Lambda_n;\beta,h}(\omega)\bigr)\bigr| \\ &= \sum_{\omega \in \Omega_{\Lambda_n}} \exp\bigl(-\mathscr{H}^{\varnothing}_{\Lambda_n;\beta,\mathfrak{Re}\, h}(\omega)\bigr) \le \exp\bigl((2d\beta + |\mathfrak{Re}\, h| + \log 2)|\Lambda_n|\bigr), \end{aligned}$$

and thus $|g_n(h)| = \exp(|\Lambda_n|^{-1} \log |\mathbf{Z}^{\varnothing}_{\Lambda_n;\beta,h}|) \le \exp(2d\beta + |\mathfrak{Re}\, h| + \log 2)$ for all $h \in D$.

We are now in a position to apply Vitali's convergence theorem (Theorem B.25) in order to conclude that $(g_n)_{n\ge1}$ converges locally uniformly, on D, to an analytic function g.

Moreover, since $g_n(h) \neq 0$ for all $h \in D$ and all $n \ge 1$, Hurwitz' theorem (Theorem B.26) implies that g has no zeroes on D. Indeed, the other possibility (that is, $g \equiv 0$ on D) is incompatible with the fact that $g = e^{\psi} > 0$ on $D \cap \mathbb{R}$.

Since g does not vanish on D, it follows from Theorem B.23 that the latter admits an analytic logarithm in D. However, choosing again the branch that is real on $D \cap \mathbb{R}$, the function $\log g$ coincides with the pressure of the Ising model on the real axis, which proves the theorem. □

To prove Theorem 3.40 using Theorem 3.42, we still have to show

Theorem 3.43 (Lee–Yang Circle Theorem)**.** *Condition* (3.51) *is satisfied when* $D = H^+$ *and when* $D = H^-$.

The proof given below will involve working with the variable

$$z \stackrel{\text{def}}{=} e^{-2h}$$

rather than h. But $h \in H^+$ if and only if $z \in \mathbb{U}$, where $\mathbb{U}$ is the open unit disk

$$\mathbb{U} \stackrel{\text{def}}{=} \{z \in \mathbb{C} : |z| < 1\}.$$

Therefore, Theorem 3.43 implies that all zeroes of $\mathbf{Z}^{\varnothing}_{\Lambda_n;\beta,h}$ (seen as a function of z) lie on the unit circle. This explains the origin of the name associated to the above result.

Proof. When $\beta = 0$, the claim is trivial. We therefore assume from now on that $\beta > 0$. It will be convenient to consider the model as defined on a subgraph of $\mathbb{Z}^d$ with no isolated vertices, that is, to consider the model on a graph (V, E) where E is a finite set of edges between nearest-neighbors of $\mathbb{Z}^d$ and where V is the set of all endpoints of edges in E. It will be assumed that the interactions among the spins on V appearing in the Hamiltonian are only between spins at vertices connected by an edge of E.

As we already said, the partition function with free boundary condition in V is a finite linear combination of powers of $e^{\pm h}$. We will now express it as a polynomial in the variable $z = e^{-2h}$. Namely,

$$\begin{aligned}
\mathbf{Z}^{\varnothing}_{V;\beta,h} &= \sum_{\omega\in\Omega_V} \prod_{\{i,j\}\in E} e^{\beta\sigma_i(\omega)\sigma_j(\omega)} \prod_{i\in V} e^{h\sigma_i(\omega)} \\
&= e^{\beta|E|+h|V|} \sum_{\omega\in\Omega_V} \prod_{\{i,j\}\in E} e^{\beta(\sigma_i(\omega)\sigma_j(\omega)-1)} \prod_{i\in V} e^{h(\sigma_i(\omega)-1)}.
\end{aligned}$$

A configuration $\omega \in \Omega_V$ can always be identified with the set $X = X(\omega) \subset V$ defined by $X(\omega) \stackrel{\text{def}}{=} \{i \in V : \sigma_i(\omega) = -1\}$. We can therefore write

$$\sum_{\omega\in\Omega_V} \prod_{\{i,j\}\in E} e^{\beta(\sigma_i(\omega)\sigma_j(\omega)-1)} \prod_{i\in V} e^{h(\sigma_i(\omega)-1)} = \sum_{X\subset V} a_E(X) z^{|X|} \stackrel{\text{def}}{=} \mathscr{P}_E(z),$$

where $a_E(\varnothing) = a_E(V) \stackrel{\text{def}}{=} 1$ and, in all other cases,

$$a_E(X) \stackrel{\text{def}}{=} \prod_{\substack{\{i,j\}\in E\\ i\in X, j\in V\setminus X}} e^{-2\beta}.$$

Observe that these coefficients satisfy $a_E(X) \in [0,1]$. Since $\mathbf{Z}^{\varnothing}_{V;\beta,h} = e^{\beta|E|+h|V|}\mathscr{P}_E(z)$, in order to show that $\mathbf{Z}^{\varnothing}_{V;\beta,h} \neq 0$ for all $h \in H^+$, it suffices to prove that $\mathscr{P}_E(z)$ does not vanish on $\mathbb{U}$.

The next step is to turn the one-variable but high-degree polynomial $\mathscr{P}_E$ into a many-variables but degree-one (in each variable) polynomial: let $\mathbf{z}_V = (z_i)_{i\in V} \in \mathbb{C}^V$ and consider the polynomial

$$\hat{\mathscr{P}}_E(\mathbf{z}_V) \stackrel{\text{def}}{=} \sum_{X\subset V} a_E(X) \prod_{i\in X} z_i.$$

Of course, the original polynomial $\mathscr{P}_E(z)$ is recovered by taking $z_i = z$ for all $i \in V$. We will show that

$$|z_i| < 1, \forall i \in V \implies \hat{\mathscr{P}}_E(\mathbf{z}_V) \neq 0\,. \tag{3.52}$$

The proof proceeds by induction on the cardinality of E. We first check that (3.52) holds when E consists of a single edge $\{i,j\}$. In that case, since $a_E(\{i\}) = a_E(\{j\}) = e^{-2\beta}$,

$$\hat{\mathscr{P}}_E(\mathbf{z}_{\{i,j\}}) = z_i z_j + e^{-2\beta}(z_i + z_j) + 1\,.$$

Therefore, $\hat{\mathscr{P}}_E(\mathbf{z}_{\{i,j\}}) = 0$ if and only if

$$z_i = -\frac{e^{-2\beta} z_j + 1}{z_j + e^{-2\beta}}\,.$$

Using the fact that $0 \le e^{-2\beta} < 1$, it is easy to check (see Exercise 3.27 below) that the Möbius transformation $z \mapsto -(e^{-2\beta}z+1)/(z+e^{-2\beta})$ interchanges the interior and the exterior of $\mathbb{U}$. This implies that if $|z_j| < 1$, then $|z_i| > 1$, so that $\hat{\mathscr{P}}_E(z_i, z_j)$ never vanishes when both $|z_i|, |z_j| < 1$.

Let us now assume that (3.52) holds for (V, E) and let $b = \{i, j\}$ be an edge of $\mathbb{Z}^d$ not contained in E. We want to show that (3.52) still holds for the graph $(V \cup \{i,j\}, E \cup \{b\})$.

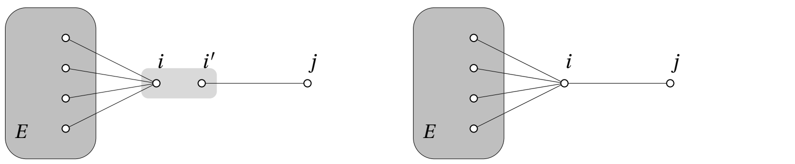
There are three cases to consider, depending on whether $V \cap \{i, j\}$ is empty, contains one vertex, or contains two vertices.

Case 1: $V \cap \{i,j\} = \varnothing$**.** In this case, the sum over $X \subset V \cup \{i,j\}$ can be split into two independent sums, over $X_1 \subset V$ and $X_2 \subset \{i,j\}$, giving

$$\hat{\mathscr{P}}_{E\cup\{b\}}(\mathbf{z}_{V\cup\{i,j\}}) = \hat{\mathscr{P}}_E(\mathbf{z}_V)\hat{\mathscr{P}}_{\{b\}}(\mathbf{z}_{\{i,j\}})\,. \tag{3.53}$$

Since neither of the polynomials on the right-hand side vanishes (by the induction hypothesis) when $|z_k| < 1$ for all $k \in V \cup \{i,j\}$, the same must be true of the polynomial on the left-hand side.

Case 2: $V \cap \{i,j\} = \{i\}$**.** The main idea here is to add the new edge b in two steps. First, we add to E a "virtual" edge $b' = \{i', j\}$, where i' is a virtual vertex not present in V, and then identify i' with i, by a procedure called Asano contraction:



On the one hand, since $V \cap \{i', j\} = \varnothing$, we are back to Case 1: the polynomial $\hat{\mathscr{P}}_{E\cup\{b'\}}(\mathbf{z}_{V\cup\{i',j\}})$ can be factorized as in (3.53) and, by the induction hypothesis, we conclude that it cannot vanish when all its variables have modulus smaller than 1.

On the other hand, the sum over $X \subset V \cup \{i', j\}$ in $\hat{\mathscr{P}}_{E\cup\{b'\}}(\mathbf{z}_{V\cup\{i',j\}})$ can be split depending on $X \cap \{i', i\}$ being $\{i, i'\}$, $\{i'\}$, $\{i\}$ or $\varnothing$, giving

$$\hat{\mathscr{P}}_{E\cup\{b'\}}(\mathbf{z}_{V\cup\{i',j\}}) = \hat{\mathscr{P}}^{-,-} z_i z_{i'} + \hat{\mathscr{P}}^{+,-} z_{i'} + \hat{\mathscr{P}}^{-,+} z_i + \hat{\mathscr{P}}^{+,+}\,,$$

where $\hat{\mathscr{P}}^{+,+}$, $\hat{\mathscr{P}}^{+,-}$, $\hat{\mathscr{P}}^{-,+}$ and $\hat{\mathscr{P}}^{-,-}$ are polynomials in the remaining variables: z_j and z_k, $k \in V \setminus \{i\}$.

The **Asano-contraction of** $\hat{\mathscr{P}}_{E\cup\{b'\}}(\mathbf{z}_{V\cup\{i',j\}})$ is defined as the polynomial

$$\hat{\mathscr{P}}^{-,-} z_i + \hat{\mathscr{P}}^{+,+} .$$

It turns out that the latter polynomial coincides with $\hat{\mathscr{P}}_{E\cup\{b\}}(\mathbf{z}_{V\cup\{j\}})$.

Lemma 3.44. $\hat{\mathscr{P}}_{E\cup\{b\}}(\mathbf{z}_{V\cup\{j\}}) = \hat{\mathscr{P}}^{-,-} z_i + \hat{\mathscr{P}}^{+,+}$.

Proof. Let $\tilde{V} \stackrel{\text{def}}{=} (V \setminus \{i\}) \cup \{j\}$. For $\sigma_1, \sigma_2 \in \{-,+\}$, the polynomials $\hat{\mathscr{P}}^{\sigma_1,\sigma_2}$ are explicitly given by

$$\hat{\mathscr{P}}^{\sigma_1,\sigma_2} = \sum_{X\subset\tilde{V}} a^{\sigma_1,\sigma_2}_{E\cup\{b'\}}(X) \prod_{k\in X} z_k ,$$

with

$$\begin{aligned}
a^{-,-}_{E\cup\{b'\}}(X) &\stackrel{\text{def}}{=} \big(\mathbf{1}_{\{X\ni j\}} + \mathbf{1}_{\{X\not\ni j\}} e^{-2\beta}\big) a_E(X\cup\{i\}) ,\\
a^{+,-}_{E\cup\{b'\}}(X) &\stackrel{\text{def}}{=} \big(\mathbf{1}_{\{X\ni j\}} + \mathbf{1}_{\{X\not\ni j\}} e^{-2\beta}\big) a_E(X) ,\\
a^{-,+}_{E\cup\{b'\}}(X) &\stackrel{\text{def}}{=} \big(\mathbf{1}_{\{X\not\ni j\}} + \mathbf{1}_{\{X\ni j\}} e^{-2\beta}\big) a_E(X\cup\{i\}) ,\\
a^{+,+}_{E\cup\{b'\}}(X) &\stackrel{\text{def}}{=} \big(\mathbf{1}_{\{X\not\ni j\}} + \mathbf{1}_{\{X\ni j\}} e^{-2\beta}\big) a_E(X) .
\end{aligned}$$

Doing a similar decomposition for the polynomial $\hat{\mathscr{P}}_{E\cup\{b\}}(\mathbf{z}_{V\cup\{j\}})$, we get:

$$\hat{\mathscr{P}}_{E\cup\{b\}}(\mathbf{z}_{V\cup\{j\}}) = \hat{\mathscr{P}}^{-} z_i + \hat{\mathscr{P}}^{+} ,$$

where, for $\sigma \in \{-,+\}$, we have introduced

$$\hat{\mathscr{P}}^{\sigma} \stackrel{\text{def}}{=} \sum_{X\subset\tilde{V}} a^{\sigma}_{E\cup\{b\}}(X) \prod_{k\in X} z_k ,$$

with

$$\begin{aligned}
a^{-}_{E\cup\{b\}}(X) &\stackrel{\text{def}}{=} \big(\mathbf{1}_{\{X\ni j\}} + \mathbf{1}_{\{X\not\ni j\}} e^{-2\beta}\big) a_E(X\cup\{i\}) ,\\
a^{+}_{E\cup\{b\}}(X) &\stackrel{\text{def}}{=} \big(\mathbf{1}_{\{X\not\ni j\}} + \mathbf{1}_{\{X\ni j\}} e^{-2\beta}\big) a_E(X) .
\end{aligned}$$

The conclusion follows. □

Since we have seen that the polynomial $\hat{\mathscr{P}}_{E\cup\{b'\}}(\mathbf{z}_{V\cup\{i',j\}})$ does not vanish when all its variables have modulus smaller than 1, it suffices to show that its Asano-contraction also cannot vanish when all its variables have modulus smaller than 1.

Let us fix the variables z_k, $k \in V \setminus \{i\}$, and z_j so that they all belong to $\mathbb{U}$. By Case 1, we know that, in this situation, $\hat{\mathscr{P}}_{E\cup\{b'\}}(\mathbf{z}_{E\cup\{i',j\}})$ cannot vanish when z_i and $z_{i'}$ also both belong to $\mathbb{U}$. By taking $z_i = z_{i'} = z$, we conclude that

$$z \mapsto \hat{\mathscr{P}}^{-,-} z^2 + (\hat{\mathscr{P}}^{-,+} + \hat{\mathscr{P}}^{+,-}) z + \hat{\mathscr{P}}^{+,+}$$

cannot have zeros of modulus smaller than 1. In particular, the product of its two roots has modulus 1 or larger. But the latter implies that $|\hat{\mathscr{P}}^{+,+}| \geq |\hat{\mathscr{P}}^{-,-}|$ and, thus, $z \mapsto \hat{\mathscr{P}}^{-,-} z + \hat{\mathscr{P}}^{+,+}$ cannot vanish when $|z| < 1$.

Case 3: $V \cap \{i, j\} = \{i, j\}$**.** This case is treated in a very similar way, so we only sketch the argument and leave the details as an exercise to the reader.

Adding a virtual edge $b'' = \{i', j'\}$ yields a polynomial $\hat{\mathscr{P}}_{E\cup\{b''\}}(\mathbf{z}_{V\cup\{i',j'\}})$ satisfying (3.52) by Case 1. We then proceed as above and apply two consecutive Asano contractions: the first to identify the variables $z_{j'}$ and z_j, the second to identify the variables $z_{i'}$ and z_i.

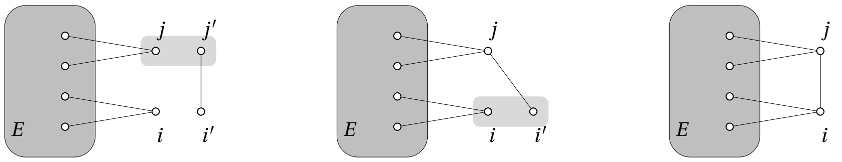


Figure 3.13: A picture of case 3: We first add a virtual edge $\{i', j'\}$ to E, then identify first j and j', and then i and i'.

□

Remark 3.45. The reader might have noticed that the proof given above does not depend on the structure of the graph inherited from the Hamiltonian of the model. Moreover, the fact that the interaction is the same between each pair of nearest-neighbor spins, was not used: the coupling constant β used for all edges could be replaced by couplings J_{ij} varying from edge to edge. Therefore, the Circle Theorem and its consequence, Theorem 3.42, can be adapted to obtain analyticity of the pressure in more general settings. ⋄

Exercise 3.27. *Let* $\varphi(z) \stackrel{\text{def}}{=} \frac{\alpha z+1}{\alpha+z}$*, where* $0 \le \alpha < 1$*. Show that* $\partial\mathbb{U}$ *is invariant under* φ*, and that* φ *maps the interior of* $\mathbb{U}$ *onto its exterior and vice versa.*

Exercise 3.28. *Using the explicit formula* (3.10) *for the pressure of the one-dimensional Ising model, determine the location of its singularities as a function of the (complex) magnetic field* h*. What happens as* β *tends to infinity?*

The next exercise provides an alternative approach to the analyticity of the pressure in a smaller open part of the complex plane, still containing $\mathbb{R} \setminus \{0\}$.

Exercise 3.29. *Assume that* $\mathfrak{Re}\, h > 0$*. Observe that, by considering two independent copies of the system with magnetic field* h *and* $\bar{h}$*, one can write*

$$|\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}|^2 = \sum_{\omega,\omega'} \exp\Big\{\beta \sum_{\{i,j\}\in\mathscr{E}_\Lambda} (\omega_i\omega_j + \omega'_i\omega'_j) + \sum_{i\in\Lambda} (h\omega_i + \bar{h}\omega'_i)\Big\}.$$

We define $\theta_i \in \{0, \pi/2, \pi, 3\pi/2\}$*,* $i \in \Lambda$*, by* $\cos\theta_i \stackrel{\text{def}}{=} \frac{1}{2}(\omega_i + \omega'_i)$ *and* $\sin\theta_i \stackrel{\text{def}}{=} \frac{1}{2}(\omega_i - \omega'_i)$*. Show that, after changing to these variables and expanding the exponential, one obtains*

$$|\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}|^2 = \sum_{(\theta_i)_{i\in\Lambda}} \sum_{\substack{\mathbf{m}=(m_i)_{i\in\Lambda}\\ m_i\in\{0,1,2,3\}}} \widehat{\alpha}_{\mathbf{m}} e^{\mathrm{i}\sum_{i\in\Lambda} m_i\theta_i} = 4^{|\Lambda|}\,\widehat{\alpha}_{(0,\dots,0)}\,,$$

with coefficients $\widehat{\alpha}_{\mathbf{m}}$ *nonnegative and nondecreasing in* $\mathfrak{Re}\, h + \mathfrak{Im}\, h$ *and in* $\mathfrak{Re}\, h - \mathfrak{Im}\, h$*. Conclude that* $|\mathbf{Z}^{\varnothing}_{\Lambda;\beta,h}| > 0$ *when* $\mathfrak{Re}\, h > |\mathfrak{Im}\, h|$*.*

3.7.5 Summary of what has been proved

In this brief subsection, we summarize the main results that have been derived. First, we emphasize that the main features of the discussion in Section 1.4.3 have been fully recovered (compare, in particular, with Figure 1.11):

Theorem 3.46. *Let $\beta_c(d)$ be the inverse critical temperature of the Ising model on $\mathbb{Z}^d$ (we have seen that $\beta_c(1) = +\infty$, while $0 < \beta_c(d) < \infty$ for $d \geq 2$).*

1. *For all $\beta < \beta_c(d)$, the average magnetization density $m(\beta, h)$ is well defined (and independent of the boundary condition and of the sequence of boxes used in its definition) for all $h \in \mathbb{R}$. It is an odd, nondecreasing, continuous function of h; in particular, $m(\beta, 0) = 0$.*
2. *For all $\beta > \beta_c(d)$, the average magnetization density $m(\beta, h)$ is well defined (and independent of the boundary condition and of the sequence of boxes used in its definition) for all $h \in \mathbb{R} \setminus \{0\}$. It is an odd, nondecreasing function of h, which is continuous everywhere except at $h = 0$, where*

$$\lim_{h \downarrow 0} m(\beta, h) = m^+(\beta, h) > 0, \quad \lim_{h \uparrow 0} m(\beta, h) = m^-(\beta, h) < 0.$$

In particular, the spontaneous magnetization satisfies

$$m^*(\beta) = 0 \text{ when } \beta < \beta_c(d), \qquad m^*(\beta) > 0 \text{ when } \beta > \beta_c(d).$$

Remark 3.47. As has already been mentioned, it is known that $m^*(\beta_c) = 0$. By Exercise 3.17, this implies that the function $\beta \mapsto m^*(\beta)$ is continuous at β_c. ⋄

Remark 3.48. It follows from the above that, when $h = 0$, the spontaneous magnetization $m^*(\beta)$ allows one to distinguish the ordered regime (in which $m^*(\beta) > 0$) from the disordered regime (in which $m^*(\beta) = 0$). A function with this property is said to be an **order parameter**. ⋄

Proof of Theorem 3.46. On the one hand, we know from Theorem 3.43 that, for all $\beta \geq 0$, the pressure $\psi(\beta, h)$ is differentiable with respect to h at all $h \neq 0$. On the other hand, point 3 of Theorem 3.25 and Theorem 3.34 imply that the function $h \mapsto \psi(\beta, h)$ is differentiable at $h = 0$ when $\beta < \beta_c(d)$, but is not differentiable at $h = 0$ when $\beta > \beta_c(d)$. This implies that $\mathfrak{B}_\beta = \varnothing$ when $h \neq 0$ or $\beta < \beta_c(d)$, and that $\mathfrak{B}_\beta = \{0\}$ when $h = 0$ and $\beta > \beta_c(d)$.

By Corollary 3.7, the above implies that $m(\beta, h)$ is well defined and independent of the boundary condition whenever $h \neq 0$ or $\beta < \beta_c(d)$. This shows, in particular, that $m(\beta, h) = m^+(\beta, h)$ for all $h > 0$.

The claim that $m(\beta, h)$ is an odd, nondecreasing function of h that is continuous for all $h \notin \mathfrak{B}_\beta$ follows from symmetry and Corollary 3.7. □

We have also seen that the Gibbs states provide a satisfactory description of the model in the thermodynamic limit. These objects give a first glimpse of the way by which models in infinite volume will be described later in the book. The states $\langle\cdot\rangle^+_{\beta,h}$ and $\langle\cdot\rangle^-_{\beta,h}$, constructed with + and − boundary conditions respectively, were instrumental in characterizing the uniqueness regime. Much more will be said on these states, in particular in Chapter 6.

3.8 Proof of the Correlation Inequalities

3.8.1 Proof of the GKS inequalities

Although the GKS inequalities (3.21) and (3.22) are already more than we need to study the (nearest-neighbor) Ising model, we will prove them in an even more general setting.

Let $\Lambda \Subset \mathbb{Z}^d$ and let $\mathbf{K} = (K_C)_{C\subset\Lambda}$ be a family of real numbers, called **coupling constants**. Consider the following probability distribution on Ω_Λ:

$$\nu_{\Lambda;\mathbf{K}}(\omega) \overset{\text{def}}{=} \frac{1}{\mathbf{Z}_{\Lambda;\mathbf{K}}} \exp\Big\{\sum_{C\subset\Lambda} K_C\omega_C\Big\},$$

where $\omega_C \overset{\text{def}}{=} \prod_{i\in C}\omega_i$ and $\mathbf{Z}_{\Lambda;\mathbf{K}}$ is the associated partition function. The Gibbs distributions $\mu^+_{\Lambda;\mathbf{J},\mathbf{h}}$, $\mu^\varnothing_{\Lambda;\mathbf{J},\mathbf{h}}$ and $\mu^{\text{per}}_{\Lambda;\mathbf{J},\mathbf{h}}$ can all be written in this form, with $K_C \ge 0$ $\forall C \subset \Lambda$, if $\mathbf{h} \ge 0$. For example, $\mu^+_{\Lambda;\beta,h} = \nu_{\Lambda;\mathbf{K}}$ once

$$K_C = \begin{cases} h + \beta\#\{j \not\in \Lambda : j \sim i\} & \text{if } C = \{i\} \subset \Lambda,\\ \beta & \text{if } C = \{i,j\} \subset \Lambda,\ i \sim j,\\ 0 & \text{otherwise.}\end{cases}$$

Exercise 3.30. *Check that $\mu^\varnothing_{\Lambda;\beta,h}$ and $\mu^{\text{per}}_{\Lambda;\beta,h}$ can also be written in this form for a suitable choice of the coefficients $\mathbf{K}$, and that these coefficients can all be taken nonnegative if $\mathbf{h} \ge 0$.*

We can now state the following generalization of Theorem 3.20.

Theorem 3.49. *Let $\mathbf{K} = (K_C)_{C\subset\Lambda}$ be such that $K_C \ge 0$ for all $C \subset \Lambda$. Then, for any $A, B \subset \Lambda$,*

$$\langle\sigma_A\rangle_{\Lambda;\mathbf{K}} \ge 0, \tag{3.54}$$

$$\langle\sigma_A\sigma_B\rangle_{\Lambda;\mathbf{K}} \ge \langle\sigma_A\rangle_{\Lambda;\mathbf{K}}\langle\sigma_B\rangle_{\Lambda;\mathbf{K}}. \tag{3.55}$$

Proof. Clearly, $\mathbf{Z}_{\Lambda;\mathbf{K}} > 0$. We can thus focus on the numerators. Expanding the exponentials as Taylor series as $e^{K_C\omega_C} = \sum_{n_C\ge 0}\frac{1}{n_C!}K_C^{n_C}\omega_C^{n_C}$, we can write

$$\begin{aligned}\mathbf{Z}_{\Lambda;\mathbf{K}}\langle\sigma_A\rangle_{\Lambda;\mathbf{K}} &= \sum_\omega \omega_A \prod_{C\subset\Lambda} e^{K_C\omega_C}\\ &= \sum_{\substack{(n_C)_{C\subset\Lambda}\\ n_C\ge 0}} \prod_{C\subset\Lambda}\frac{K_C^{n_C}}{n_C!}\sum_\omega \omega_A\prod_{C\subset\Lambda}\omega_C^{n_C}.\end{aligned} \tag{3.56}$$

We rewrite $\omega_A\prod_{C\subset\Lambda}\omega_C^{n_C} = \prod_{i\in\Lambda}\omega_i^{m_i}$, where $m_i = \mathbf{1}_{\{i\in A\}} + \sum_{C\subset\Lambda, C\ni i} n_C$. Upon summation, since

$$\sum_{\omega_i=\pm1}\omega_i^{m_i} = \begin{cases}2 & \text{if } m_i \text{ is even,}\\ 0 & \text{if } m_i \text{ is odd,}\end{cases}$$

it follows that

$$\sum_\omega\prod_{i\in\Lambda}\omega_i^{m_i} = \prod_{i\in\Lambda}\sum_{\omega_i=\pm1}\omega_i^{m_i} \ge 0.$$

This establishes (3.54). To prove (3.55), we duplicate the system. That is, we consider the product probability distribution $\nu_{\Lambda;\mathbf{K}} \otimes \nu_{\Lambda;\mathbf{K}}$ on $\Omega_\Lambda \times \Omega_\Lambda$ defined by

$$\nu_{\Lambda;\mathbf{K}} \otimes \nu_{\Lambda;\mathbf{K}}(\omega,\omega') \stackrel{\text{def}}{=} \nu_{\Lambda;\mathbf{K}}(\omega)\nu_{\Lambda;\mathbf{K}}(\omega').$$

If we define $\sigma_i(\omega,\omega') \stackrel{\text{def}}{=} \omega_i$ and $\sigma'_i(\omega,\omega') \stackrel{\text{def}}{=} \omega'_i$, then

$$\langle \sigma_A \sigma_B \rangle_{\Lambda;\mathbf{K}} - \langle \sigma_A \rangle_{\Lambda;\mathbf{K}} \langle \sigma_B \rangle_{\Lambda;\mathbf{K}} = \langle \sigma_A(\sigma_B - \sigma'_B)\rangle_{\nu_{\Lambda;\mathbf{K}}\otimes\nu_{\Lambda;\mathbf{K}}}.$$

The problem is thus reduced to proving the nonnegativity of

$$(\mathbf{Z}_{\Lambda;\mathbf{K}})^2 \langle \sigma_A(\sigma_B - \sigma'_B)\rangle_{\nu_{\Lambda;\mathbf{K}}\otimes\nu_{\Lambda;\mathbf{K}}} = \sum_{\omega,\omega'} \omega_A(\omega_B - \omega'_B) \prod_{C\subset\Lambda} e^{K_C(\omega_C + \omega'_C)}.$$

Introducing the variables $\omega''_i \stackrel{\text{def}}{=} \omega_i \omega'_i = \omega'_i/\omega_i$,

$$\begin{aligned}\sum_{\omega,\omega'} \omega_A(\omega_B - \omega'_B) \prod_{C\subset\Lambda} e^{K_C(\omega_C+\omega'_C)} &= \sum_{\omega,\omega''} \omega_A\omega_B(1-\omega''_B) \prod_{C\subset\Lambda} e^{K_C(1+\omega''_C)\omega_C} \\ &= \sum_{\omega''}(1-\omega''_B)\sum_{\omega}\omega_A\omega_B\prod_{C\subset\Lambda} e^{K_C(1+\omega''_C)\omega_C}.\end{aligned}$$

Since $1-\omega''_B \geq 0$, (3.55) follows by treating this last sum over ω (for each fixed ω'') as the one in (3.56), working with coupling constants $K_C(1+\omega''_C) \geq 0$. □

Exercise 3.31. *Let* $\mathbf{K} = (K_C)_{C\subset\Lambda}$ *and* $\mathbf{K}' = (K'_C)_{C\subset\Lambda}$ *be such that* $K_C \geq |K'_C|$ *(in particular,* $K_C \geq 0$*), for all* $C \subset \Lambda$*. Show that, for any* $A, B \subset \Lambda$*,*

$$\langle \sigma_A \rangle_{\Lambda;\mathbf{K}} \geq \langle \sigma_A \rangle_{\Lambda;\mathbf{K}'}.$$

Hint: *apply a variant of the argument used to prove* (3.55).

3.8.2 Proof of the FKG inequality

We provide here a very general and short proof of the FKG inequality. The interested reader can find an alternative proof in Section 3.10.3, based on Markov chain techniques, which he might find more intuitive.

Our aim is to show that, for a finite volume $\Lambda \Subset \mathbb{Z}^d$ and two nondecreasing functions $f, g : \Omega \to \mathbb{R}$,

$$\langle fg \rangle^\eta_{\Lambda;\mathbf{J},\mathbf{h}} \geq \langle f \rangle^\eta_{\Lambda;\mathbf{J},\mathbf{h}} \langle g \rangle^\eta_{\Lambda;\mathbf{J},\mathbf{h}}. \tag{3.57}$$

Again, we will prove a result that is more general than required. Remember that the order we use on Ω_Λ is the following: $\omega \leq \omega'$ if and only if $\omega_i \leq \omega'_i$ for all $i \in \Lambda$. We also define, for $\omega = (\omega_i)_{i\in\Lambda}$ and $\omega' = (\omega'_i)_{i\in\Lambda}$,

$$\omega \wedge \omega' \stackrel{\text{def}}{=} (\omega_i \wedge \omega'_i)_{i\in\Lambda},$$
$$\omega \vee \omega' \stackrel{\text{def}}{=} (\omega_i \vee \omega'_i)_{i\in\Lambda}.$$

As explained below, (3.57) is a consequence of the following general result.

Theorem 3.50. *Let $\mu = \bigotimes_{i\in\Lambda}\mu_i$ be a product measure on Ω_Λ. Let $f_1,\dots,f_4 : \Omega_\Lambda \to \mathbb{R}$ be nonnegative functions on Ω_Λ such that*

$$f_1(\omega)f_2(\omega') \le f_3(\omega\wedge\omega')f_4(\omega\vee\omega'), \qquad \forall \omega,\omega' \in \Omega_\Lambda. \tag{3.58}$$

Then

$$\langle f_1\rangle_\mu \langle f_2\rangle_\mu \le \langle f_3\rangle_\mu \langle f_4\rangle_\mu. \tag{3.59}$$

Before turning to the proof of this result, let us explain why it implies (3.57). With no loss of generality, we can assume that f and g depend only on the values of the configuration inside Λ and that both are nonnegative [5]. For $i \in \Lambda$, $s \in \{\pm1\}$, let

$$\mu_i(s) \stackrel{\text{def}}{=} e^{hs+s\sum_{j\notin\Lambda, j\sim i} J_{ij}\eta_j}.$$

We have

$$\langle f\rangle^\eta_{\Lambda;\mathbf{J},\mathbf{h}} = \sum_{\omega\in\Omega_\Lambda} f(\omega)p(\omega)\mu(\omega) = \langle fp\rangle_\mu,$$

where

$$p(\omega) \stackrel{\text{def}}{=} \frac{\exp\{\sum_{\{i,j\}\in\mathscr{E}_\Lambda} J_{ij}\omega_i\omega_j\}}{\mathbf{Z}^\eta_{\Lambda;\mathbf{J},\mathbf{h}}}.$$

Let $f_1 = pf$, $f_2 = pg$, $f_3 = p$, $f_4 = pfg$. If (3.58) holds for this choice, then (3.59) holds, and so (3.57) is proved. To check (3.58), we must verify that

$$p(\omega)p(\omega') \le p(\omega\vee\omega')p(\omega\wedge\omega').$$

But this is true since

$$\omega_i\omega_j + \omega'_i\omega'_j \le (\omega_i\vee\omega'_i)(\omega_j\vee\omega'_j) + (\omega_i\wedge\omega'_i)(\omega_j\wedge\omega'_j).$$

Indeed, the inequality is obvious if both terms in the right-hand side are equal to 1. Let us therefore assume that at least one of them is equal to -1. This cannot happen if both $\omega_i \ne \omega'_i$ and $\omega_j \ne \omega'_j$. Without loss of generality, we can thus suppose that $\omega_i = \omega'_i$. In that case, the right-hand side equals

$$\omega_i\big\{(\omega_j\vee\omega'_j) + (\omega_j\wedge\omega'_j)\big\} = \omega_i(\omega_j+\omega'_j) = \omega_i\omega_j + \omega'_i\omega'_j.$$

Remark 3.51. As the reader can easily check, the proof below does not rely on the fact that the spins take their values in $\{\pm1\}$; it actually holds for arbitrary real-valued spins. ⋄

Proof of Theorem 3.50. For some fixed $i \in \Lambda$, any configuration $\omega \in \Omega_\Lambda$ can be identified with the pair $(\tilde\omega, \omega_i)$, where $\tilde\omega \in \Omega_{\Lambda\setminus\{i\}}$. We will show that

$$f_1(\omega)f_2(\omega') \le f_3(\omega\wedge\omega')f_4(\omega\vee\omega'). \tag{3.60}$$

implies

$$\tilde f_1(\tilde\omega)\tilde f_2(\tilde\omega') \le \tilde f_3(\tilde\omega\wedge\tilde\omega')\tilde f_4(\tilde\omega\vee\tilde\omega'), \tag{3.61}$$

[5] Indeed, if these hypotheses are not verified, we can redefine $f(\omega)$, for $\omega \in \Omega_\Lambda$, by $f(\omega\eta|_{\Lambda^c}) - \min_{\omega'} f(\omega'\eta|_{\Lambda^c})$ where $\omega\eta|_{\Lambda^c}$ is the configuration that coincides with ω on Λ and with η on Λ^c. The same can be done with g. Note that this does not affect the covariance of f and g.

where (for $k = 1,2,3,4$) $\tilde{f}_k(\tilde{\omega}) \stackrel{\text{def}}{=} \langle f_k(\tilde{\omega},\cdot)\rangle_{\mu_i} = \sum_{v=\pm 1} f_k(\tilde{\omega},v)\mu_i(v)$. Using this observation $|\Lambda|$ times yields the desired result.

The left-hand side of (3.61) can be written

$$\langle f_1(\tilde{\omega},u)f_2(\tilde{\omega}',v)\rangle_{\mu_i\otimes\mu_i} = \langle \mathbf{1}_{\{u=v\}} f_1(\tilde{\omega},u)f_2(\tilde{\omega}',v)\rangle_{\mu_i\otimes\mu_i} + \langle \mathbf{1}_{\{u<v\}}(f_1(\tilde{\omega},u)f_2(\tilde{\omega}',v) + f_1(\tilde{\omega},v)f_2(\tilde{\omega}',u))\rangle_{\mu_i\otimes\mu_i}.$$

Similarly, the right-hand side of (3.61) can be written

$$\langle f_3(\tilde{\omega}\wedge\tilde{\omega}',u)f_4(\tilde{\omega}\vee\tilde{\omega}',v)\rangle_{\mu_i\otimes\mu_i} = \langle \mathbf{1}_{\{u=v\}} f_3(\tilde{\omega}\wedge\tilde{\omega}',u)f_4(\tilde{\omega}\vee\tilde{\omega}',v)\rangle_{\mu_i\otimes\mu_i} + \langle \mathbf{1}_{\{u<v\}}(f_3(\tilde{\omega}\wedge\tilde{\omega}',u)f_4(\tilde{\omega}\vee\tilde{\omega}',v) + f_3(\tilde{\omega}\wedge\tilde{\omega}',v)f_4(\tilde{\omega}\vee\tilde{\omega}',u))\rangle_{\mu_i\otimes\mu_i}.$$

We thus obtain

$$\begin{aligned}\tilde{f}_3(\tilde{\omega}\wedge\tilde{\omega}')\tilde{f}_4(\tilde{\omega}\vee\tilde{\omega}') - \tilde{f}_1(\tilde{\omega})\tilde{f}_2(\tilde{\omega}') &= \langle \mathbf{1}_{\{u=v\}}\big(f_3(\tilde{\omega}\wedge\tilde{\omega}',u)f_4(\tilde{\omega}\vee\tilde{\omega}',v) - f_1(\tilde{\omega},u)f_2(\tilde{\omega}',v)\big)\rangle_{\mu_n\otimes\mu_n}\\ &\quad + \langle \mathbf{1}_{\{u<v\}}\big(C+D-A-B\big)\rangle_{\mu_n\otimes\mu_n}, \end{aligned} \tag{3.62}$$

where we have introduced $A \stackrel{\text{def}}{=} f_1(\tilde{\omega},u)f_2(\tilde{\omega}',v)$, $B \stackrel{\text{def}}{=} f_1(\tilde{\omega},v)f_2(\tilde{\omega}',u)$, $C \stackrel{\text{def}}{=} f_3(\tilde{\omega}\wedge\tilde{\omega}',u)f_4(\tilde{\omega}\vee\tilde{\omega}',v)$ and $D \stackrel{\text{def}}{=} f_3(\tilde{\omega}\wedge\tilde{\omega}',v)f_4(\tilde{\omega}\vee\tilde{\omega}',u)$.

The first term in the right-hand side of (3.62) is nonnegative thanks to inequality (3.60). The desired claim (3.61) will thus follow if we can show that $A+B \leq C+D$.

Observe first that (3.60) implies that $A \leq C$, $B \leq C$ and

$$\begin{aligned} AB &= f_1(\tilde{\omega},u)f_2(\tilde{\omega}',u)\, f_1(\tilde{\omega},v)f_2(\tilde{\omega}',v)\\ &\leq f_3(\tilde{\omega}\wedge\tilde{\omega}',u)f_4(\tilde{\omega}\vee\tilde{\omega}',u)\, f_3(\tilde{\omega}\wedge\tilde{\omega}',v)f_4(\tilde{\omega}\vee\tilde{\omega}',v) = CD. \end{aligned}$$

On the one hand, if $C = 0$, then $A = B = 0$ and the inequality $A+B \leq C+D$ is obvious. On the other hand, when $C \neq 0$, the inequality follows from

$$(C+D-A-B)/C \geq 1 + AB/C^2 - (A+B)/C = (1-A/C)(1-B/C) \geq 0. \qquad \square$$

3.9 Bibliographical references

The Ising model is probably the most studied model in statistical physics and, as such, is discussed in countless books and review articles. An old, but very good, general discussion in the spirit of what is done here is [146]. We list some references for the material presented in the chapter.

Pressure. The notion of convergence in the sense of van Hove (formulated in a slightly different, but equivalent way) was first introduced in [345].

In the context of lattice spin systems, the existence and the basic properties of the thermodynamic limit for the pressure were first established by Griffiths [145] and Gallavotti and Miracle-Solé [128]. The proofs given in this chapter (Theorem 3.6 and Exercise 3.3) can be extended to cover a very wide class of models, possibly with interactions of infinite range. See the books by Ruelle [289] and Simon [308] for additional results and information.

The computation of the pressure of the one-dimensional (nearest-neighbor) Ising model (Theorem 3.9) was the main result of Ising's PhD thesis and was published in [175]. It relied on some simple combinatorics in order to compute the generating function $\sum_N \mathbf{Z}_{V_N;\beta,h} s^N$, from which Ising then extracted the value of the partition function. The transfer matrix computation seems to be due to Kramers and Wannier [200].

The first computation of the pressure of the two-dimensional Ising model without magnetic field, whose result is stated at the end of Section 3.3, was achieved in a groundbreaking work by Onsager [259]. Extensions of the computations to nonzero magnetic field in two dimensions, or to higher dimensions, have not been found despite much effort.

Gibbs states. The notion of Gibbs state as used in this chapter (rather than the more general version discussed in Chapter 6) was commonly used in the 1960s and 1970s, see, for example, the early review by Gallavotti [127].

Correlation inequalities and applications. The first version of the GKS inequalities was obtained by Griffiths [142]; in the form stated in Theorem 3.49 they are due to Kelly and Sherman [188]. These inequalities admit important generalizations to more general single-spin spaces; see, for example, [139, 310]. The proof of the GKS inequalities given in Section 3.8 is due to Ginibre [138].

The FKG inequality has first been established by Fortuin, Kasteleyn and Ginibre [110]. The proof given in Section 3.8.2 is due to Ahlswede and Daykin [2]; our presentation is inspired by [10]. The alternative proof presented in Section 3.10.3 was found by Holley [163]; see also [132, 225].

The applications of the correlation inequalities given in Section 3.6 are part of the folklore and are spread out over many papers. A good early reference is [146]. Exercise 3.15 is adapted from [229].

The uniqueness criteria given in Theorems 3.28 and 3.34 are due to Lebowitz and Martin-Löf [219]. The other claims concerning the magnetization density are again part of the folklore.

Peierls' argument. The geometric proof described in Section 3.7.2 is due to Peierls [266]; see also [144, 80]. This argument has become central in the rigorous analysis of first-order phase transitions and is at the basis of the Pirogov–Sinai theory, a far-reaching generalization which is the main topic of Chapter 7.

The approach described in Exercise 3.21 is inspired by [198]. The more precise bounds on the connectivity constant $2.625622 < \mu < 2.679193$ can be found in [182] and [277] respectively. Numerically, the best estimate at the moment of writing seems to be $\mu \cong 2.63815853032790(3)$ [180].

High-temperature representation. The high-temperature representation, which is described in Section 3.7.3, was introduced by van der Waerden in [340].

The proof of uniqueness based on the high-temperature expansion is again part of the folklore. There are many alternative ways of establishing uniqueness at high enough temperature, among which: Dobrushin's uniqueness theorem (discussed in Section 6.5.2), the cluster expansion (discussed in Section 6.5.4) and disagreement percolation (see, for example, [132]). These can be used to extract additional information, such as analyticity of the pressure, exponential decay of correlations, exponential convergence of the finite-volume expectations of local functions, etc.; see [86] for a discussion of the remarkable additional properties that hold at sufficiently high temperatures.

Uniqueness in non-zero magnetic field. Theorems 3.40, 3.42 and 3.43 are due to Lee and Yang and appeared first in [353, 220]. The Asano contraction method used in the proof of the latter theorem was introduced by Asano in [13]; see also [290]. For a rather extensive bibliography on this topic and various extensions, see [33].

Although we do not discuss this in the text, it is possible to derive various properties of interest from the Lee–Yang theorem, such as exponential decay of truncated correlation functions (for example, $\langle\sigma_0\sigma_i\rangle_{\beta,h} - \langle\sigma_0\rangle_{\beta,h}\langle\sigma_i\rangle_{\beta,h}$) at all β when $h \neq 0$ [95], as well as analyticity in h of correlation functions [216]. See also [120, 121].

Another route to the proof of uniqueness at non-zero magnetic field is through the GHS inequality. The latter was first proved by Griffiths, Hurst and Sherman in [143]. It states that the Ising model with magnetic field $\mathbf{h} = (h_i)_{i\in\Lambda}$ satisfies

$$\frac{\partial^2}{\partial h_i \partial h_j}\langle\sigma_k\rangle^{\varnothing}_{\Lambda;\beta,\mathbf{h}} \leq 0\,,$$

for all $\Lambda \Subset \mathbb{Z}^d$ and $i, j, k \in \Lambda$, provided that $h_\ell \geq 0$ for all $\ell \in \Lambda$. Taking $h_i = h$ for all i, it implies in particular that the magnetization density $m(\beta, h)$ is concave (in particular, continuous) as a function of $h \geq 0$.

The alternative argument given in Exercise 3.29 is adapted from a more general approach by Dunlop [96].

3.10 Complements and further reading

3.10.1 Kramers–Wannier duality

In this section we present an argument, proposed by Kramers and Wannier [200], which suggests that the critical inverse temperature of the Ising model on $\mathbb{Z}^2$ is equal to

$$\beta_c(2) = \tfrac{1}{2}\log(1+\sqrt{2})\,. \tag{3.63}$$

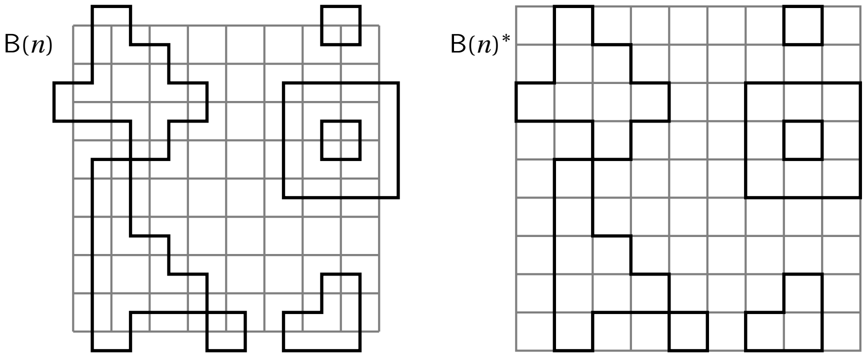
The starting point is the representation of the partition function with + boundary condition in terms of contours in (3.32):

$$\mathbf{Z}^+_{\mathsf{B}(n);\beta,0} = e^{\beta|\mathscr{E}^{\mathrm{b}}_{\mathsf{B}(n)}|} \sum_{\omega\in\Omega^+_{\mathsf{B}(n)}} \prod_{\gamma\in\Gamma(\omega)} e^{-2\beta|\gamma|}\,. \tag{3.64}$$

Let $\mathsf{B}(n)^* = \{-n-\frac{1}{2}, -n+\frac{1}{2}, \dots, n-\frac{1}{2}, n+\frac{1}{2}\}^2 \subset \mathbb{Z}^2_*$ be the box dual to $\mathsf{B}(n)$. From Exercise 3.23, we have the high-temperature representation

$$\mathbf{Z}^{\varnothing}_{\mathsf{B}(n)^*;\beta^*,0} = 2^{|\mathsf{B}(n)^*|}\cosh(\beta^*)^{|\mathscr{E}_{\mathsf{B}(n)^*}|} \sum_{E\in\mathfrak{C}^{\mathrm{even}}_{\mathsf{B}(n)^*}} \tanh(\beta^*)^{|E|}\,. \tag{3.65}$$

We will now identify each set $E \in \mathfrak{C}^{\mathrm{even}}_{\mathsf{B}(n)^*}$ with the edges of the contours of a unique configuration $\omega \in \Omega^+_{\mathsf{B}(n)}$:



Lemma 3.52. *Let $E \in \mathscr{E}_{\mathsf{B}(n)^*}$. Then $E \in \mathfrak{E}^{\text{even}}_{\mathsf{B}(n)^*}$ if and only if E coincides with the edges of the contours of a configuration $\omega \in \Omega^+_{\mathsf{B}(n)}$.*

Proof. If $E \in \mathfrak{E}^{\text{even}}_{\mathsf{B}(n)^*}$, then applying the rounding operation of Figure 3.11 yields a set of disjoint closed loops which are the contours of the configuration $\omega \in \Omega^+_{\mathsf{B}(n)}$ defined by

$$\omega_i \stackrel{\text{def}}{=} (-1)^{|\{\text{loops surrounding } i\}|}, \quad i \in \mathsf{B}(n).$$

Conversely, we have already seen in footnote 2, page 111, that the set of edges of the contours of a configuration $\omega \in \Omega^+_{\mathsf{B}(n)}$ belong to $\mathfrak{E}^{\text{even}}_{\mathsf{B}(n)^*}$. □

It follows from the previous lemma that

$$\sum_{E \in \mathfrak{E}^{\text{even}}_{\mathsf{B}(n)^*}} \tanh(\beta^*)^{|E|} = \sum_{\omega \in \Omega^+_{\mathsf{B}(n)}} \prod_{\gamma \in \Gamma(\omega)} \tanh(\beta^*)^{|\gamma|}.$$

Therefore, if β^* satisfies

$$\tanh(\beta^*) = e^{-2\beta}, \tag{3.66}$$

we obtain the identity

$$2^{-|\mathsf{B}(n)^*|} \cosh(\beta^*)^{-|\mathscr{E}_{\mathsf{B}(n)^*}|} \mathbf{Z}^{\varnothing}_{\mathsf{B}(n)^*;\beta^*,0} = e^{-\beta|\mathscr{E}^{\mathrm{b}}_{\mathsf{B}(n)}|} \mathbf{Z}^+_{\mathsf{B}(n);\beta,0}. \tag{3.67}$$

When $n \to \infty$,

$$\frac{|\mathsf{B}(n)^*|}{|\mathsf{B}(n)|} \to 1, \quad \frac{|\mathscr{E}_{\mathsf{B}(n)^*}|}{|\mathsf{B}(n)|} \to 2, \quad \frac{|\mathscr{E}^{\mathrm{b}}_{\mathsf{B}(n)}|}{|\mathsf{B}(n)|} \to 2.$$

We thus obtain, by Theorem 3.6,

$$\psi(\beta, 0) = \psi(\beta^*, 0) - \log\sinh(2\beta^*). \tag{3.68}$$

The meaning of (3.68) is that the pressure is essentially invariant under the transformation

$$\beta \mapsto \beta^* = \operatorname{arctanh}(e^{-2\beta}), \tag{3.69}$$

which interchanges the low and high temperatures, as can be verified in the following exercise.

Exercise 3.32. *Show that the mapping $\phi : x \mapsto \operatorname{arctanh}(e^{-2x})$ is an involution ($\phi \circ \phi = \mathrm{id}$) with a unique fixed (self-dual) point β_{sd} equal to $\frac{1}{2}\log(1+\sqrt{2})$. Moreover, $\phi([0, \beta_{\mathrm{sd}})) = (\beta_{\mathrm{sd}}, \infty]$.*

Since ϕ and $\log\sinh$ are both analytic on $(0,\infty)$, it follows from (3.68) that any non-analytic behavior of $\psi(\cdot,0)$ at some inverse temperature β must also imply a non-analytic behavior at $\beta^* = \phi(\beta)$. Consequently, if one assumes that the pressure $\psi(\cdot,0)$

1. is non-analytic at β_c,
2. is analytic everywhere else,

then β_c must coincide with β_{sd}. This leads to the conjecture (3.63).

That the inverse critical temperature of the Ising model on $\mathbb{Z}^2$ actually coincides with the self-dual point of this transformation follows from the exact expression for the pressure derived by Onsager. There exists in fact a variety of ways to prove that this is the correct value for β_c in the two-dimensional Ising model, relying on the self-duality of the model, but avoiding exact computations; see, for example, [350]. Extensions to other planar graphs is possible, see [70] and references therein.

The duality relation (3.67) and various generalizations have found numerous other uses in the rigorous analysis of the two-dimensional Ising model. The book by Gruber, Hintermann and Merlini [154] discusses duality in considerably more detail and in a more general framework.

3.10.2 Mean-field bounds

Let $\psi_\beta^{\mathrm{CW}}(h)$, $m_\beta^{\mathrm{CW}}(h)$ and $\beta_c^{\mathrm{CW}} \stackrel{\mathrm{def}}{=} (2d)^{-1}$ be the pressure, magnetization and critical inverse temperature of the Curie–Weiss model associated to the d-dimensional Ising model (remember the dependence on d in the Hamiltonian (2.2)). The following theorem, due to Thompson [332, 330], shows that these quantities provide rigorous bounds on the corresponding quantities for the Ising model on $\mathbb{Z}^d$. References to additional results pertaining to the relations between a model on $\mathbb{Z}^d$ and its mean-field approximation can be found in Section 2.5.4.

Theorem 3.53. *The following holds for the Ising model on $\mathbb{Z}^d$, $d \geq 1$:*

1. $\psi(\beta,h) \geq \psi_\beta^{\mathrm{CW}}(h)$, *for all $\beta \geq 0$ and all $h \in \mathbb{R}$;*
2. $\langle\sigma_0\rangle_{\beta,h}^+ \leq m_\beta^{\mathrm{CW}}(h)$, *for all $\beta \geq 0$ and all $h \geq 0$;*
3. $\beta_c(d) \geq \beta_c^{\mathrm{CW}}$, *for all $d \geq 1$.*

Proof. 1. Since the pressures are even functions of h, we can assume that $h \geq 0$. We start by decomposing the Hamiltonian with periodic boundary condition:

$$\mathscr{H}_{V_n;\beta,h}^{\mathrm{per}} \stackrel{\mathrm{def}}{=} -\beta \sum_{\{i,j\}\in\mathscr{E}_{V_n}^{\mathrm{per}}} \sigma_i\sigma_j - h\sum_{i\in V_n}\sigma_i = \mathscr{H}_{V_n;\beta,h}^{\mathrm{per},0} + \mathscr{H}_{V_n;\beta,h}^{\mathrm{per},1},$$

where

$$\mathscr{H}_{V_n;\beta,h}^{\mathrm{per},0} \stackrel{\mathrm{def}}{=} d\beta|V_n|m^2 - (h+2d\beta m)\sum_{i\in V_n}\sigma_i\,,$$

$$\mathscr{H}_{V_n;\beta,h}^{\mathrm{per},1} \stackrel{\mathrm{def}}{=} -\beta \sum_{\{i,j\}\in\mathscr{E}_{V_n}^{\mathrm{per}}} (\sigma_i - m)(\sigma_j - m)\,,$$

Revised version, August 22 2017
To be published by Cambridge University Press (2017)

www.unige.ch/math/folks/velenik/smbook

where $m \in \mathbb{R}$ will be chosen later. We can then rewrite the corresponding partition function as

$$\begin{aligned}
\mathbf{Z}^{\mathrm{per}}_{V_n;\beta,h} &\overset{\mathrm{def}}{=} \sum_{\omega\in\Omega_{V_n}} \exp\bigl(-\mathscr{H}^{\mathrm{per}}_{V_n;\beta,h}(\omega)\bigr) \\
&= \sum_{\omega\in\Omega_{V_n}} \exp\bigl(-\mathscr{H}^{\mathrm{per},1}_{V_n;\beta,h}(\omega)\bigr)\exp\bigl(-\mathscr{H}^{\mathrm{per},0}_{V_n;\beta,h}(\omega)\bigr) \\
&= \mathbf{Z}^{\mathrm{per},0}_{V_n;\beta,h}\,\bigl\langle \exp\bigl(-\mathscr{H}^{\mathrm{per},1}_{V_n;\beta,h}\bigr)\bigr\rangle^{\mathrm{per},0}_{V_n;\beta,h},
\end{aligned}$$

where we have introduced the Gibbs distribution

$$\mu^{\mathrm{per},0}_{V_n;\beta,h}(\omega) \overset{\mathrm{def}}{=} \frac{\exp\bigl(-\mathscr{H}^{\mathrm{per},0}_{V_n;\beta,h}(\omega)\bigr)}{\mathbf{Z}^{\mathrm{per},0}_{V_n;\beta,h}}, \qquad \text{with} \qquad \mathbf{Z}^{\mathrm{per},0}_{V_n;\beta,h} \overset{\mathrm{def}}{=} \sum_{\omega\in\Omega_{V_n}} \exp\bigl(-\mathscr{H}^{\mathrm{per},0}_{V_n;\beta,h}(\omega)\bigr).$$

By Jensen's inequality,

$$\mathbf{Z}^{\mathrm{per}}_{V_n;\beta,h} \geq \mathbf{Z}^{\mathrm{per},0}_{V_n;\beta,h}\,\exp\bigl(-\bigl\langle \mathscr{H}^{\mathrm{per},1}_{V_n;\beta,h}\bigr\rangle^{\mathrm{per},0}_{V_n;\beta,h}\bigr).$$

Observe that

$$\begin{aligned}
\bigl\langle \mathscr{H}^{\mathrm{per},1}_{V_n;\beta,h}\bigr\rangle^{\mathrm{per},0}_{V_n;\beta,h} &= -\beta \sum_{\{i,j\}\in\mathscr{E}^{\mathrm{per}}_{V_n}} \bigl(\langle\sigma_i\rangle^{\mathrm{per},0}_{V_n;\beta,h} - m\bigr)\bigl(\langle\sigma_j\rangle^{\mathrm{per},0}_{V_n;\beta,h} - m\bigr) \\
&= -\beta d|V_n|\bigl(m - \langle\sigma_0\rangle^{\mathrm{per},0}_{V_n;\beta,h}\bigr)^2.
\end{aligned}$$

Since

$$\langle\sigma_0\rangle^{\mathrm{per},0}_{V_n;\beta,h} = \tanh(2d\beta m + h),$$

choosing m to be the largest solution to

$$m = \tanh(2d\beta m + h)$$

we get $\bigl\langle \mathscr{H}^{\mathrm{per},1}_{V_n;\beta,h}\bigr\rangle^{\mathrm{per},0}_{V_n;\beta,h} = 0$ and, therefore,

$$\mathbf{Z}^{\mathrm{per}}_{V_n;\beta,h} \geq \mathbf{Z}^{\mathrm{per},0}_{V_n;\beta,h} = e^{-d\beta m^2|V_n|}2^{|V_n|}\cosh(2d\beta m + h)^{|V_n|}.$$

The conclusion follows (just compare with the expression in Exercise 2.4).

2. Let $\Lambda = \mathsf{B}(n)$, with $n \geq 1$, and let $i \sim 0$ be any nearest-neighbor of the origin. Let $\langle\cdot\rangle^{+,1}_{\Lambda;\beta,h}$ denote the expectation with respect to the Gibbs distribution in Λ with no interaction between the two vertices 0 and i. Then, using (3.41),

$$\begin{aligned}
\langle\sigma_0\rangle^{+}_{\Lambda;\beta,h} &= \frac{\sum_{\omega\in\Omega^+_\Lambda}\omega_0\exp\bigl\{\beta\sum_{\{j,k\}\in\mathscr{E}^{\mathrm{b}}_\Lambda\setminus\{0,i\}}\omega_j\omega_k\bigr\}\bigl(1+\omega_0\omega_i\tanh\beta\bigr)}{\sum_{\omega\in\Omega^+_\Lambda}\exp\bigl\{\beta\sum_{\{j,k\}\in\mathscr{E}^{\mathrm{b}}_\Lambda\setminus\{0,i\}}\omega_j\omega_k\bigr\}\bigl(1+\omega_0\omega_i\tanh\beta\bigr)} \\
&= \frac{\langle\sigma_0\rangle^{+,1}_{\Lambda;\beta,h} + \langle\sigma_i\rangle^{+,1}_{\Lambda;\beta,h}\tanh\beta}{1+\langle\sigma_0\sigma_i\rangle^{+,1}_{\Lambda;\beta,h}\tanh\beta} \\
&\leq \frac{\langle\sigma_0\rangle^{+,1}_{\Lambda;\beta,h} + \langle\sigma_i\rangle^{+,1}_{\Lambda;\beta,h}\tanh\beta}{1+\langle\sigma_0\rangle^{+,1}_{\Lambda;\beta,h}\langle\sigma_i\rangle^{+,1}_{\Lambda;\beta,h}\tanh\beta}, \qquad (3.70)
\end{aligned}$$

where we used the GKS inequality. Now, observe that, for any $x \ge 0$, $a \in [0,1]$ and $b \in [-1,1]$,

$$\frac{b + a\tanh(x)}{1 + ba\tanh(x)} \le \frac{b + \tanh(ax)}{1 + b\tanh(ax)}. \tag{3.71}$$

Indeed, $y \mapsto (b+y)/(1+by)$ is increasing in $y \ge 0$, and $\tanh(ax) \ge a\tanh(x)$ (by concavity). Applying (3.71) to (3.70), we get

$$\langle\sigma_0\rangle^+_{\Lambda;\beta,h} \le \frac{\langle\sigma_0\rangle^{+,1}_{\Lambda;\beta,h} + \tanh\bigl(\beta\langle\sigma_i\rangle^{+,1}_{\Lambda;\beta,h}\bigr)}{1 + \langle\sigma_0\rangle^{+,1}_{\Lambda;\beta,h}\tanh\bigl(\beta\langle\sigma_i\rangle^{+,1}_{\Lambda;\beta,h}\bigr)}.$$

But, since $\bigl(\tanh(x) + \tanh(y)\bigr) / \bigl(1 + \tanh(x)\tanh(y)\bigr) = \tanh(x+y)$, this gives

$$\langle\sigma_0\rangle^+_{\Lambda;\beta,h} \le \tanh\bigl\{\operatorname{arctanh}\bigl(\langle\sigma_0\rangle^{+,1}_{\Lambda;\beta,h}\bigr) + \beta\langle\sigma_i\rangle^{+,1}_{\Lambda;\beta,h}\bigr\},$$

which can be rewritten as

$$\operatorname{arctanh}\bigl(\langle\sigma_0\rangle^+_{\Lambda;\beta,h}\bigr) \le \operatorname{arctanh}\bigl(\langle\sigma_0\rangle^{+,1}_{\Lambda;\beta,h}\bigr) + \beta\langle\sigma_i\rangle^{+,1}_{\Lambda;\beta,h}.$$

Finally, by GKS inequalities, $\langle\sigma_i\rangle^{+,1}_{\Lambda;\beta,h} \le \langle\sigma_i\rangle^+_{\Lambda;\beta,h}$, so that

$$\operatorname{arctanh}\bigl(\langle\sigma_0\rangle^+_{\Lambda;\beta,h}\bigr) \le \operatorname{arctanh}\bigl(\langle\sigma_0\rangle^{+,1}_{\Lambda;\beta,h}\bigr) + \beta\langle\sigma_i\rangle^+_{\Lambda;\beta,h}. \tag{3.72}$$

Clearly, one can iterate (3.72), removing all edges between 0 and its nearest-neighbors, one at a time. This yields

$$\operatorname{arctanh}\bigl(\langle\sigma_0\rangle^+_{\Lambda;\beta,h}\bigr) \le \operatorname{arctanh}\bigl(\langle\sigma_0\rangle^{\varnothing}_{\{0\};\beta,h}\bigr) + \beta\sum_{i\sim 0}\langle\sigma_i\rangle^+_{\Lambda;\beta,h}.$$

Of course, $\langle\sigma_0\rangle^{\varnothing}_{\{0\};\beta,h} = \tanh(h)$. Therefore,

$$\operatorname{arctanh}\bigl(\langle\sigma_0\rangle^+_{\Lambda;\beta,h}\bigr) \le h + \beta\sum_{i\sim 0}\langle\sigma_i\rangle^+_{\Lambda;\beta,h},$$

that is,

$$\langle\sigma_0\rangle^+_{\Lambda;\beta,h} \le \tanh\bigl(h + \beta\sum_{i\sim 0}\langle\sigma_i\rangle^+_{\Lambda;\beta,h}\bigr).$$

We can now let $\Lambda \uparrow \mathbb{Z}^d$ and use the fact that $\langle\sigma_i\rangle^+_{\beta,h} = \langle\sigma_0\rangle^+_{\beta,h}$ for all i to obtain the desired bound:

$$\langle\sigma_0\rangle^+_{\beta,h} \le \tanh\bigl(h + 2d\beta\langle\sigma_0\rangle^+_{\beta,h}\bigr).$$

From this we conclude that $\langle\sigma_0\rangle^+_{\beta,h} \le m^{\mathrm{CW}}_\beta(h)$.

3. When $\beta < \beta^{\mathrm{CW}}_{\mathrm{c}}$, the previous item implies that $\langle\sigma_0\rangle^+_{\beta,0} \le m^{\mathrm{CW}}_\beta(0) = 0$. This implies $\beta < \beta_{\mathrm{c}}(d)$, which proves the claim. □

3.10.3 An alternative proof of the FKG inequality

Here, we provide an alternative proof of the FKG inequality. Although possibly less general and somewhat longer than the one provided in Section 3.8.2, we believe that it has the undeniable advantage of being more enlightening. It relies on some basic knowledge of discrete-time finite-state Markov chains, as exposed, for example, in the book [156].

The Gibbs sampler. Let $\Lambda \Subset \mathbb{Z}^d$ and let μ be some probability distribution on $\Omega_\Lambda = \{-1,1\}^\Lambda$ satisfying $\mu(\omega) > 0$ for all $\omega \in \Omega_\Lambda$.

We construct a discrete-time Markov chain $(X_n)_{n\geq 0}$ on Ω_Λ as follows: given that $X_n = \omega \in \Omega_\Lambda$, the value of X_{n+1}, say ω', is sampled using the following algorithm:

1. Sample a number u according to the uniform distribution on $[0,1]$ (independently of all other sources of randomness).
2. Sample a vertex $i \in \Lambda$ with uniform distribution (independently of all other sources of randomness).
3. Set $\omega'_j = \omega_j$ for all $j \neq i$.
4. Set
$$\omega'_i = \begin{cases} +1 & \text{if } u \leq \mu\big(\sigma_i = 1 \mid \sigma_j = \omega_j \ \forall j \neq i\big), \\ -1 & \text{otherwise.} \end{cases}$$

In other words, there are no transitions between two configurations differing at more than one vertex; moreover, given two configurations $\omega, \omega' \in \Omega_\Lambda$ differing at a single vertex $i \in \Lambda$, the transition probability from ω to ω' is given by

$$p(\omega \to \omega') = \frac{1}{|\Lambda|}\,\mu\big(\sigma_i = \omega'_i \mid \sigma_j = \omega_j \ \forall j \neq i\big) = \frac{1}{|\Lambda|}\,\frac{\mu(\omega')}{\mu(\omega) + \mu(\omega')}\,.$$

Observe that the Markov chain $(X_n)_{n\geq 0}$ is irreducible (since one can move between two arbitrary configurations by changing one spin at a time, each such transition occurring with positive probability) and aperiodic (since $p(\omega \to \omega) > 0$). Therefore the distribution of X_n converges almost surely towards the unique stationary distribution. We claim that the latter is given by μ. Indeed, $(X_n)_{n\geq 0}$ is reversible with respect to μ: if $\omega, \omega' \in \Omega_\Lambda$ are two configurations differing only at one vertex, then

$$\mu(\omega)\,p(\omega \to \omega') = \frac{1}{|\Lambda|}\,\frac{\mu(\omega)\mu(\omega')}{\mu(\omega) + \mu(\omega')} = \mu(\omega')\,p(\omega' \to \omega)\,.$$

Monotone coupling. Let us now consider two probability distributions μ and $\tilde{\mu}$ on Ω_Λ. As above, we assume that $\mu(\omega) > 0$ and $\tilde{\mu}(\omega) > 0$. Moreover, we assume that

$$\mu\big(\sigma_i = 1 \mid \sigma_j = \omega_j \ \forall j \neq i\big) \leq \tilde{\mu}\big(\sigma_i = 1 \mid \sigma_j = \tilde{\omega}_j \ \forall j \neq i\big)\,, \tag{3.73}$$

for all $\omega, \tilde{\omega} \in \Omega_\Lambda$ such that $\tilde{\omega} \geq \omega$.

Let us denote by $(X_n)_{n\geq 0}$ and $(\tilde{X}_n)_{n\geq 0}$ the Markov chains on Ω_Λ associated to μ and $\tilde{\mu}$, as described above. We are going to define the **monotone coupling** of these two Markov chains. The coupling is defined by the previous construction, but using, at each step of the process, the *same* $u \in [0,1]$ and $i \in \Lambda$ for both chains. The important observation is that

$$\tilde{X}_n \geq X_n \implies \tilde{X}_{n+1} \geq X_{n+1}\,.$$

Indeed, let us denote by i the vertex which has been selected at this step. In order to violate the inequality $\tilde{X}_{n+1} \geq X_{n+1}$, it is necessary that $\sigma_i(X_{n+1}) = 1$ and $\sigma_i(\tilde{X}_{n+1}) = -1$. But this is impossible, since for the former to be true, one needs to have $u \leq \mu(\sigma_i = 1 \,|\, \sigma_j = \sigma_j(X_n) \ \forall j \neq i)$, which, by (3.73), would imply that $u \leq \tilde{\mu}(\sigma_i = 1 \,|\, \sigma_j = \sigma_j(\tilde{X}_n) \ \forall j \neq i)$ and, thus, $\sigma_i(\tilde{X}_{n+1}) = 1$.

Stochastic domination. Let μ and $\tilde{\mu}$ be as above. It is now very easy to prove that, for every nondecreasing function f,

$$\langle f\rangle_{\tilde{\mu}} \geq \langle f\rangle_{\mu}\,. \tag{3.74}$$

In that case, we say that **$\tilde{\mu}$ stochastically dominates μ**.

Let us consider the two monotonically coupled Markov chains, as described above, with initial values $X_0 = \eta^- \equiv -1$ and $\tilde{X}_0 = \eta^+ \equiv 1$. We denote by $\mathbb{P}$ the distribution of the coupled Markov chains. Now, since these chains converge, respectively, to μ and $\tilde{\mu}$, we can write

$$\langle f\rangle_{\tilde{\mu}} - \langle f\rangle_{\mu} = \lim_{n\to\infty} \sum_{\eta,\tilde{\eta}\in\Omega_\Lambda} \bigl\{f(\tilde{\eta}) - f(\eta)\bigr\}\,\mathbb{P}\bigl(X_n = \eta, \tilde{X}_n = \tilde{\eta}\bigr)\,.$$

Moreover, by monotonicity of the coupling,

$$\mathbb{P}\bigl(\tilde{X}_n \geq X_n, \text{ for all } n \geq 0\bigr) = 1\,.$$

We can thus restrict the summation to pairs $\tilde{\eta} \geq \eta$:

$$\langle f\rangle_{\tilde{\mu}} - \langle f\rangle_{\mu} = \lim_{n\to\infty} \sum_{\substack{\eta,\tilde{\eta}\in\Omega_\Lambda\\ \tilde{\eta}\geq\eta}} \bigl\{f(\tilde{\eta}) - f(\eta)\bigr\}\,\mathbb{P}\bigl(X_n = \eta, \tilde{X}_n = \tilde{\eta}\bigr)\,.$$

(3.74) follows since $\tilde{\eta} \geq \eta$ implies that $f(\tilde{\eta}) - f(\eta) \geq 0$.

Proof of the FKG inequality. We can now prove the FKG inequality for the Ising model on $\mathbb{Z}^d$. Let $\Lambda \Subset \mathbb{Z}^d$, $\eta \in \Omega$, $\beta \geq 0$ and $h \in \mathbb{R}$. We want to prove that

$$\langle fg\rangle^{\eta}_{\Lambda;\beta,h} \geq \langle f\rangle^{\eta}_{\Lambda;\beta,h}\,\langle g\rangle^{\eta}_{\Lambda;\beta,h}\,, \tag{3.75}$$

for all nondecreasing functions f and g. Note that we can, and will, assume that $g(\tau) > 0$ for all $\tau \in \Omega^{\eta}_{\Lambda}$, since adding a constant to g does not affect (3.75). We can thus consider the following two probability distributions on Ω_Λ:

$$\mu(\omega) \stackrel{\text{def}}{=} \mu^{\eta}_{\Lambda;\beta,h}(\omega\eta)\,, \qquad \tilde{\mu}(\omega) \stackrel{\text{def}}{=} \frac{g(\omega\eta)}{\langle g\rangle^{\eta}_{\Lambda;\beta,h}}\,\mu^{\eta}_{\Lambda;\beta,h}(\omega\eta)\,,$$

where, given $\omega \in \Omega_\Lambda$, $\omega\eta$ denotes the configuration coinciding with ω in Λ and with η outside Λ. Clearly $\mu(\omega) > 0$ and $\tilde{\mu}(\omega) > 0$ for all $\omega \in \Omega_\Lambda$. (3.75) can then rewritten as

$$\langle f\rangle_{\tilde{\mu}} \geq \langle f\rangle_{\mu}\,.$$

Since this is exactly (3.74), it is sufficient to prove that (3.73) holds for these two distributions.

Observe first that, since g is nondecreasing,

$$\begin{aligned}
\tilde{\mu}\bigl(\sigma_i = 1 \bigm| \sigma_j = \tilde{\omega}_j\ \forall j \neq i\bigr) &= \frac{\mu((+1)\tilde{\omega})g((+1)\tilde{\omega})}{\mu((+1)\tilde{\omega})g((+1)\tilde{\omega}) + \mu((-1)\tilde{\omega})g((-1)\tilde{\omega})}\\
&= \Bigl\{1 + \frac{\mu((-1)\tilde{\omega})}{\mu((+1)\tilde{\omega})}\,\frac{g((-1)\tilde{\omega})}{g((+1)\tilde{\omega})}\Bigr\}^{-1}\\
&\geq \Bigl\{1 + \frac{\mu((-1)\tilde{\omega})}{\mu((+1)\tilde{\omega})}\Bigr\}^{-1},
\end{aligned}$$

where $(+1)\tilde\omega$, resp. $(-1)\tilde\omega$, is the configuration given by $\tilde\omega$ at vertices different from i and by $+1$, resp. -1, at i.

Now,

$$\frac{\mu((-1)\tilde\omega)}{\mu((+1)\tilde\omega)} = \frac{\mu^\eta_{\Lambda;\beta,h}((-1)\tilde\omega\eta)}{\mu^\eta_{\Lambda;\beta,h}((+1)\tilde\omega\eta)} = \exp\Bigl(-2\beta\sum_{j\sim i}(\tilde\omega\eta)_j - 2h\Bigr)$$

is a nonincreasing function of $\tilde\omega$. It follows that, for any $\omega\in\Omega_\Lambda$ such that $\tilde\omega\geq\omega$,

$$\begin{aligned}\tilde\mu(\sigma_i = 1 \,|\, \sigma_j = \tilde\omega_j\ \forall j\neq i) &\geq \Bigl\{1+\frac{\mu((-1)\omega)}{\mu((+1)\omega)}\Bigr\}^{-1}\\ &= \mu\bigl(\sigma_i = 1 \bigm| \sigma_j = \omega_j\ \forall j\neq i\bigr),\end{aligned}$$

and (3.73), and thus (3.75), follows.

3.10.4 Transfer matrix and Markov chains

In Section 3.3, we described how the pressure of the one-dimensional Ising model could be determined using the transfer matrix. Readers familiar with Markov chains might have noted certain obvious similarities. In this complement, we explain how these tools can be related and what additional information can be extracted.

Let A be the transfer matrix of the one-dimensional Ising model, defined in (3.11). For simplicity, let us denote by $\mathbf{Z}_n^{s,s'} \equiv \mathbf{Z}^{\eta^{s,s'}}_{\Lambda_n;\beta,h}$, $s,s'\in\{\pm1\}$, the partition function of the model on $\Lambda_n = \{1,\dots,n\}$, with boundary condition $\eta^{s,s'}$ given by $\eta_i^{s,s'} = s$ if $i\leq 0$ and $\eta_i^{s,s'} = s'$ if $i>0$.

Proceeding as in Section 3.3, the transfer matrix can be related to the partition function $\mathbf{Z}_n^{s,s'}$ in the following way: for all $n\geq1$,

$$\mathbf{Z}_n^{s,s'} = \bigl(A^{n+1}\bigr)_{s,s'}\,.$$

Let $\lambda>0$ be the largest of the two eigenvalues of A. We denote by φ, respectively φ^*, the right-eigenvector, respectively left-eigenvector, associated to λ: $A\varphi=\lambda\varphi$, $\varphi^*A = \lambda\varphi^*$. We assume that these eigenvectors satisfy the following normalization assumption: $\varphi\cdot\varphi^* = 1$. All these quantities can be computed explicitly, but we will not need the resulting expressions here. Notice however that, either by an explicit computation or by the Perron–Frobenius theorem [45, Theorem 1.1], all components of φ and φ^* are positive.

We now define a new matrix $\Pi = (\pi_{s,s'})_{s,s'=\pm1}$ by

$$\pi_{s,s'} \overset{\text{def}}{=} \frac{\varphi_{s'}}{\lambda\varphi_s}A_{s,s'}\,.$$

Π is the transition matrix of an irreducible, aperiodic Markov chain. Indeed, for $s\in\{\pm1\}$,

$$\sum_{s'\in\{\pm1\}}\pi_{s,s'} = \frac{1}{\lambda\varphi_s}\sum_{s'\in\{\pm1\}}A_{s,s'}\varphi_{s'} = \frac{1}{\lambda\varphi_s}\bigl(A\varphi\bigr)_s = 1\,.$$

Irreducibility and aperiodicity follow from the positivity of $\pi_{s,s'}$ for all $s,s'\in\{\pm1\}$.

Being irreducible, Π possesses a unique stationary distribution ν, given by

$$\nu(\{s\}) = \varphi_s\varphi^*_s\,,\qquad s\in\{\pm1\}\,.$$

Revised version, August 22 2017
To be published by Cambridge University Press (2017)

www.unige.ch/math/folks/velenik/smbook

Indeed, $\nu(\{1\}) + \nu(\{-1\}) = 1$, by our normalization assumption, and

$$\big(\nu\Pi\big)(\{s'\}) = \sum_{s\in\{\pm1\}} \nu(\{s\})\,\pi_{s,s'} = \frac{1}{\lambda}\varphi_{s'} \sum_{s\in\{\pm1\}} \varphi^*_s\, A_{s,s'} = \varphi_{s'}\varphi^*_{s'} = \nu(\{s'\})\,,$$

since $\varphi^* A = \lambda\varphi^*$.

The probability distribution ν on $\{\pm1\}$ provides the distribution of σ_0 under the infinite-volume Gibbs state. Indeed, denoting by $\mu^{s,s'}_{\mathsf{B}(n);\beta,h}$ the Gibbs distribution on $\mathsf{B}(n) = \{-n,\dots,n\}$ with boundary condition $\eta^{s,s'}$, the probability that $\sigma_0 = s_0$ is given by

$$\mu^{s,s'}_{\mathsf{B}(n);\beta,h}(\sigma_0 = s_0) = \frac{\mathbf{Z}_n^{s,s_0}\mathbf{Z}_n^{s_0,s'}}{\mathbf{Z}^{s,s'}_{2n+1}} = \frac{\big(A^{n+1}\big)_{s,s_0}\big(A^{n+1}\big)_{s_0,s'}}{\big(A^{2n+2}\big)_{s,s'}}\,.$$

Now, as can be checked, for any $s, s' \in \{\pm1\}$,

$$\big(A^n\big)_{s,s'} = \lambda^n \frac{\varphi_s}{\varphi_{s'}}\big(\Pi^n\big)_{s,s'}\,,$$

which gives, after substitution in the above expression,

$$\mu^{s,s'}_{\mathsf{B}(n);\beta,h}(\sigma_0 = s_0) = \frac{\big(\Pi^n\big)_{s,s_0}\big(\Pi^n\big)_{s_0,s'}}{\big(\Pi^{2n+2}\big)_{s,s'}}\,.$$

Since the Markov chain is irreducible and aperiodic, $\lim_{n\to\infty}(\Pi^n)_{s,s'} = \nu(\{s'\})$ for all $s, s' \in \{\pm1\}$. We conclude that

$$\lim_{n\to\infty}\mu^{s,s'}_{\mathsf{B}(n);\beta,h}(\sigma_0 = s_0) = \frac{\nu(\{s_0\})\nu(\{s'\})}{\nu(\{s'\})} = \nu(\{s_0\})\,.$$

One can check similarly that the joint distribution of any finite collection $(\sigma_i)_{a\le i\le b}$ of spins is given by

$$\lim_{n\to\infty}\mu^{s,s'}_{\mathsf{B}(n);\beta,h}\big(\sigma_k = s_k\,, \forall a \le k \le b\big) = \nu(\{s_a\}) \prod_{k=a}^{b-1} \pi_{s_k,s_{k+1}}\,.$$

The interested reader can find much more information, in a more general setting, in [134, Chapter 11].

3.10.5 The Ising antiferromagnet

The **Ising antiferromagnet** is a model whose neighboring spins tend to point in *opposite* directions, this effect becoming stronger at lower temperatures. It therefore does not exhibit spontaneous magnetization.

We only consider the antiferromagnet in the absence of a magnetic field. This model can be thought of as an Ising model with negative coupling constants:

$$\mathscr{H}^{\text{anti}}_{\Lambda;\beta}(\omega) \stackrel{\text{def}}{=} \beta \sum_{\{i,j\}\in\mathscr{E}^{\text{b}}_\Lambda} \sigma_i(\omega)\sigma_j(\omega)\,. \tag{3.76}$$

Let a vertex $i = (i_1,\dots,i_d) \in \mathbb{Z}^d$ be called **even** (resp. **odd**) if $i_1 + \cdots + i_d$ is even (resp. odd). Consider the transformation $\tau_{\text{even}} : \Omega \to \Omega$ defined by

$$(\tau_{\text{even}}\omega)_i \stackrel{\text{def}}{=} \begin{cases} +\omega_i & \text{if } i \text{ is even,} \\ -\omega_i & \text{otherwise.} \end{cases}$$

One can then define $\tau_{\text{odd}} : \Omega \to \Omega$ by

$$(\tau_{\text{odd}}\omega)_i \stackrel{\text{def}}{=} -(\tau_{\text{even}}\omega)_i\,, \qquad i \in \mathbb{Z}^d\,.$$

Not surprisingly, the main features of this model can be derived from the results obtained for the Ising model:

Exercise 3.33. *Observing that*

$$\mathscr{H}^{\text{anti}}_{\Lambda;\beta}(\omega) = \mathscr{H}_{\Lambda;\beta}(\tau_{\text{even}}\omega)\,,$$

use the results obtained in this chapter to show that, when $\beta > \beta_c(d)$, two distinct Gibbs states can be constructed, $\langle\cdot\rangle^{\text{even}}_\beta$ and $\langle\cdot\rangle^{\text{odd}}_\beta$. Describe the typical configurations under these two states.

Let us just emphasize that the trick used in the previous exercise to reduce the analysis to the ferromagnetic case relies in an essential way on the fact that the lattice $\mathbb{Z}^d$ is **bipartite**, that is, one can color each of its vertices in either black or white in such a way that no neighboring vertices have the same color. On a non-bipartite lattice, or in the presence of a magnetic field, the behavior of the antiferromagnet is much more complicated; some aspects will be discussed in Exercises 7.5 and 7.7.

3.10.6 Random-cluster and random-current representations.

In this chapter, we chose an approach to the Ising model that we deemed best suited to the generalization to other models done in the rest of the book. In particular, we barely touched on the topics of *geometrical representations*: we only introduced the low- and high-temperature representations in Sections 3.7.2 and 3.7.3, in the course of our analysis of the phase diagram. In this section, we briefly introduce two other graphical representations that have played and continue to play a central role in the mathematical analysis of the Ising model, the *random-cluster* and *random-current representations.*

Good references to the random-cluster representation can be found in the review paper [132] by Georgii, Häggström and Maes, and the books by Grimmett [150] and Werner [350]. The lecture notes [91] by Duminil-Copin provide a good introduction to several graphical representations, including the random-cluster and random-current representations. In addition to the latter, graphical representations of correlation functions in terms of interacting random paths (an example of which being the high-temperature representation of Section 3.7.3) are also very important tools; a thorough discussion can be found in the book [102] by Fernández, Fröhlich and Sokal.

The random-cluster representation. This representation was introduced by Fortuin and Kasteleyn [109]. Besides playing an instrumental role in many mathematical investigations of the Ising model, it also provides a deep link with other classical models, in particular the q-state Potts model and the Bernoulli bond percolation process. Moreover, this representation is the basis of numerical algorithms, first introduced by Swendsen and Wang [323], that are very efficient at sampling from such Gibbs distributions.

The starting point is similar to what was done to derive the high-temperature representation of the model: we expand in a suitable way the Boltzmann weight.

Here, we write

$$e^{\beta\sigma_i\sigma_j} = e^{-\beta} + (e^{\beta} - e^{-\beta})\mathbf{1}_{\{\sigma_i=\sigma_j\}} = e^{\beta}\big((1-p_\beta) + p_\beta \mathbf{1}_{\{\sigma_i=\sigma_j\}}\big),$$

where we have introduced $p_\beta \stackrel{\text{def}}{=} 1 - e^{-2\beta} \in [0,1]$.

Let $\Lambda \Subset \mathbb{Z}^d$. Using the above notations, we obtain, after expanding the product (remember Exercise 3.22),

$$\prod_{\{i,j\}\in\mathscr{E}_\Lambda^{\text{b}}} e^{\beta\sigma_i\sigma_j} = e^{\beta|\mathscr{E}_\Lambda^{\text{b}}|} \sum_{E\subset\mathscr{E}_\Lambda^{\text{b}}} p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}_\Lambda^{\text{b}}\setminus E|} \prod_{\{i,j\}\in E} \mathbf{1}_{\{\sigma_i=\sigma_j\}}.$$

The partition function $\mathbf{Z}^+_{\Lambda;\beta,0}$ can thus be expressed as

$$\begin{aligned}\mathbf{Z}^+_{\Lambda;\beta,0} &= e^{\beta|\mathscr{E}_\Lambda^{\text{b}}|} \sum_{E\subset\mathscr{E}_\Lambda^{\text{b}}} p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}_\Lambda^{\text{b}}\setminus E|} \sum_{\omega\in\Omega_\Lambda^+} \prod_{\{i,j\}\in E} \mathbf{1}_{\{\sigma_i(\omega)=\sigma_j(\omega)\}}\\ &= e^{\beta|\mathscr{E}_\Lambda^{\text{b}}|} \sum_{E\subset\mathscr{E}_\Lambda^{\text{b}}} p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}_\Lambda^{\text{b}}\setminus E|}\, 2^{N_\Lambda^{\text{w}}(E)-1},\end{aligned}$$

where $N_\Lambda^{\text{w}}(E)$ denotes the number of connected components (usually called **clusters** in this context) of the graph $(\mathbb{Z}^d, E\cup\mathscr{E}_{\mathbb{Z}^d\setminus\Lambda})$ (in other words, the graph obtained by considering all vertices of $\mathbb{Z}^d$ and all edges of $\mathbb{Z}^d$ which either belong to E or do not intersect the box Λ). Indeed, in the sum over $\omega\in\Omega_\Lambda^+$, the only configurations contributing are those in which all spins belonging to the same cluster agree.

The **FK-percolation process** in Λ with **wired boundary condition** is the probability distribution on the set $\mathscr{P}(\mathscr{E}_\Lambda^{\text{b}})$ of all subsets of $\mathscr{E}_\Lambda^{\text{b}}$ assigning to a subset of edges $E\subset\mathscr{E}_\Lambda^{\text{b}}$ the probability

$$\nu^{\text{FK,w}}_{\Lambda;p_\beta,2}(E) \stackrel{\text{def}}{=} \frac{p_\beta^{|E|}(1-p_\beta)^{|\mathscr{E}_\Lambda^{\text{b}}\setminus E|}\, 2^{N_\Lambda^{\text{w}}(E)}}{\sum_{E'\subset\mathscr{E}_\Lambda^{\text{b}}} p_\beta^{|E'|}(1-p_\beta)^{|\mathscr{E}_\Lambda^{\text{b}}\setminus E'|}\, 2^{N_\Lambda^{\text{w}}(E')}}.$$

Remark 3.54. Observe that, by replacing the factor 2 in the above expression by 1, the distribution $\nu^{\text{FK,w}}_{\Lambda;p_\beta,2}$ reduces to the Bernoulli bond percolation process on $\mathscr{E}_\Lambda^{\text{b}}$, in which each edge of $\mathscr{E}_{\mathbb{Z}^d}$ belongs to E with probability p_β, independently from the other edges. Similarly, the random-cluster representation of the q-state Potts model is obtained by replacing the factor 2 by q. In this sense, the FK-percolation process provides a one-parameter family of models interpolating between Bernoulli percolation, Ising and Potts models. ⋄

For $A,B\subset\mathbb{Z}^d$, let us write $\{A\leftrightarrow B\}$ for the event that there exists a cluster intersecting both A and B.

Exercise 3.34. *Proceeding as above, check the following identities: for any $i,j\in\Lambda\Subset\mathbb{Z}^d$,*

$$\langle\sigma_i\rangle^+_{\Lambda;\beta,0} = \nu^{\text{FK,w}}_{\Lambda;p_\beta,2}(i\leftrightarrow\partial^{\text{ex}}\Lambda), \qquad \langle\sigma_i\sigma_j\rangle^+_{\Lambda;\beta,0} = \nu^{\text{FK,w}}_{\Lambda;p_\beta,2}(i\leftrightarrow j).$$

One feature that makes the random-cluster representation particularly useful, as it makes it possible to successfully import many ideas and techniques developed for Bernoulli bond percolation, is the availability of an FKG inequality. Let $\Lambda\Subset\mathbb{Z}^d$ and consider the partial order on $\mathscr{P}(\mathscr{E}_\Lambda^{\text{b}})$ given by $E\le E'$ if and only if $E\subset E'$.

Exercise 3.35. *Show, using Theorem 3.50, that*

$$\nu^{\mathrm{FK,w}}_{\Lambda;p_\beta,2}(\mathscr{A}\cap\mathscr{B}) \geq \nu^{\mathrm{FK,w}}_{\Lambda;p_\beta,2}(\mathscr{A})\,\nu^{\mathrm{FK,w}}_{\Lambda;p_\beta,2}(\mathscr{B}),$$

for all pairs $\mathscr{A},\mathscr{B}$ of nondecreasing events on $\mathscr{P}(\mathscr{E}^{\mathrm{b}}_\Lambda)$.

As an immediate application, one can prove the existence of the thermodynamic limit.

Exercise 3.36. *Show that, for every local increasing event $\mathscr{A}$,*

$$\lim_{\Lambda\uparrow\mathbb{Z}^d} \nu^{\mathrm{FK,w}}_{\Lambda;p_\beta,2}(\mathscr{A})$$

exists. Hint: *proceed as in the proof of Theorem 3.17.*

As already mentioned in Remark 3.15 and as will be explained in more detail in Chapter 6, it follows from the previous exercise and the Riesz–Markov–Kakutani representation theorem that one can define a probability measure $\nu^{\mathrm{FK,w}}_{p_\beta,2}$ on $\mathscr{E}$ such that

$$\nu^{\mathrm{FK,w}}_{p_\beta,2}(\mathscr{A}) = \lim_{\Lambda\uparrow\mathbb{Z}^d} \nu^{\mathrm{FK,w}}_{\Lambda;p_\beta,2}(\mathscr{A}),$$

for all local events $\mathscr{A}$. A simple but remarkable observation is that the statements of Exercise 3.34 still hold under $\nu^{\mathrm{FK,w}}_{p_\beta,2}$. In particular,

$$\langle\sigma_0\rangle^+_{\beta,0} = \nu^{\mathrm{FK,w}}_{p_\beta,2}(0\leftrightarrow\infty), \tag{3.77}$$

where $\{0\leftrightarrow\infty\} \stackrel{\text{def}}{=} \bigcap_n\{0\leftrightarrow\partial^{\mathrm{ex}}\mathrm{B}(n)\}$ corresponds to the event that there exists an infinite path of disjoint open edges starting from 0 (or, equivalently, that the cluster containing 0 has infinite cardinality). Since Theorem 3.28 shows that the existence of a first-order phase transition at inverse temperature β (and magnetic field $h=0$) is equivalent to $\langle\sigma_0\rangle^+_{\beta,0}>0$, the above relation implies that the latter is also equivalent to **percolation** in the associated FK-percolation process. This observation provides new insights into the phase transition we have studied in this chapter and provides the basis for a geometrical analysis of the Ising model using methods inherited from percolation theory.

Exercise 3.37. *Prove the identity* (3.77).

The random-current representation. Also of great importance in the mathematical analysis of the Ising model, with many fundamental applications, this representation had already been introduced in [143], but its true power was realized by Aizenman [4].

Once again, the strategy is to expand the Boltzmann weight in a suitable way, then expand the product over pairs of neighbors, and finally sum explicitly over the spins. For the first step, we simply expand the exponential as a Taylor series:

$$e^{\beta\sigma_i\sigma_j} = \sum_{n=0}^{\infty}\frac{\beta^n}{n!}(\sigma_i\sigma_j)^n.$$

We then get, writing $\mathbf{n} = (n_e)_{e\in\mathscr{E}^{\mathrm{b}}_\Lambda}$ for a collection of nonnegative integers,

$$\prod_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda} e^{\beta\sigma_i\sigma_j} = \sum_{\mathbf{n}} \Big\{ \prod_{e\in\mathscr{E}^{\mathrm{b}}_\Lambda} \frac{\beta^{n_e}}{n_e!} \Big\} \prod_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda} (\sigma_i\sigma_j)^{n_{\{i,j\}}} .$$

The partition function $\mathbf{Z}^+_{\Lambda;\beta,0}$ can thus be expressed as

$$\begin{aligned}\mathbf{Z}^+_{\Lambda;\beta,0} &= \sum_{\mathbf{n}} \Big\{ \prod_{e\in\mathscr{E}^{\mathrm{b}}_\Lambda} \frac{\beta^{n_e}}{n_e!} \Big\} \sum_{\omega\in\Omega^+_\Lambda} \prod_{\{i,j\}\in\mathscr{E}^{\mathrm{b}}_\Lambda} (\sigma_i(\omega)\sigma_j(\omega))^{n_{\{i,j\}}} \\ &= \sum_{\mathbf{n}} \Big\{ \prod_{e\in\mathscr{E}^{\mathrm{b}}_\Lambda} \frac{\beta^{n_e}}{n_e!} \Big\} \prod_{i\in\Lambda} \sum_{\omega_i=\pm1} \omega_i^{\hat{I}(i,\mathbf{n})} ,\end{aligned}$$

where $\hat{I}(i,\mathbf{n}) \stackrel{\text{def}}{=} \sum_{j:\,j\sim i} n_{\{i,j\}}$. Since

$$\sum_{\omega_i=\pm1} \omega_i^m = \begin{cases} 2 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd,}\end{cases}$$

we conclude that

$$\mathbf{Z}^+_{\Lambda;\beta,0} = 2^{|\Lambda|} \sum_{\mathbf{n}:\partial_\Lambda\mathbf{n}=\varnothing} \prod_{e\in\mathscr{E}^{\mathrm{b}}_\Lambda} \frac{\beta^{n_e}}{n_e!} = 2^{|\Lambda|} e^{\beta|\mathscr{E}^{\mathrm{b}}_\Lambda|} \mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda \mathbf{n} = \varnothing) ,$$

where $\partial_\Lambda\mathbf{n} \stackrel{\text{def}}{=} \big\{ i\in\Lambda : \hat{I}(i,\mathbf{n}) \text{ is odd} \big\}$ and, under the probability distribution $\mathbb{P}^+_{\Lambda;\beta}$, $\mathbf{n} = (n_e)_{e\in\mathscr{E}^{\mathrm{b}}_\Lambda}$ is a collection of independent random variables, each one distributed according to the Poisson distribution of parameter β. We will call $\mathbf{n}$ a **current configuration** in Λ.

In the same way, one easily derives similar representations for arbitrary correlation functions.

Exercise 3.38. *Derive the following identity: for all $A\subset\Lambda\Subset\mathbb{Z}^d$,*

$$\langle\sigma_A\rangle^+_{\Lambda;\beta,0} = \frac{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda\mathbf{n} = A)}{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda\mathbf{n} = \varnothing)} .$$

The power of the random-current representation, however, lies in the fact that it also allows a probabilistic interpretation of *truncated correlations* in terms of various geometric events. The crucial result is the following lemma, which deals with a distribution on *pairs* of current configurations $\mathbb{P}^{+(2)}_{\Lambda;\beta}(\mathbf{n}^1,\mathbf{n}^2) \stackrel{\text{def}}{=} \mathbb{P}^+_{\Lambda;\beta}(\mathbf{n}^1)\mathbb{P}^+_{\Lambda;\beta}(\mathbf{n}^2)$. Let us denote by $i \stackrel{\mathbf{n}}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda$ the event that there is a path connecting i to $\partial^{\mathrm{ex}}\Lambda$ along which $\mathbf{n}$ takes only positive values.

Lemma 3.55 (Switching Lemma)**.** *Let $\Lambda\Subset\mathbb{Z}^d$, $A\subset\Lambda$, $i\in\Lambda$ and $\mathscr{I}$ a set of current configurations in Λ. Then,*

$$\begin{aligned}&\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}^1 = A, \partial_\Lambda\mathbf{n}^2 = \{i\}, \mathbf{n}^1+\mathbf{n}^2\in\mathscr{I}) \\ &\qquad = \mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda\mathbf{n}^1 = A\,\triangle\,\{i\}, \partial_\Lambda\mathbf{n}^2 = \varnothing, \mathbf{n}^1+\mathbf{n}^2\in\mathscr{I}, i \stackrel{\mathbf{n}^1+\mathbf{n}^2}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda) . \qquad (3.78)\end{aligned}$$

Revised version, August 22 2017
To be published by Cambridge University Press (2017)

www.unige.ch/math/folks/velenik/smbook

Proof. We will use the following notations:

$$\mathrm{w}(\mathbf{n}) \stackrel{\text{def}}{=} \prod_{e\in\mathscr{E}_\Lambda} \frac{\beta^{n_e}}{n_e!}$$

and, for two current configurations satisfying $\mathbf{n} \le \mathbf{m}$ (that is, $n_e \le m_e$, $\forall e \in \mathscr{E}_\Lambda^{\mathrm{b}}$),

$$\binom{\mathbf{m}}{\mathbf{n}} \stackrel{\text{def}}{=} \prod_{e\in\mathscr{E}_\Lambda^{\mathrm{b}}} \binom{m_e}{n_e}.$$

We are going to change variables from the pair $(\mathbf{n}^1,\mathbf{n}^2)$ to the pair $(\mathbf{m},\mathbf{n})$ where $\mathbf{m} = \mathbf{n}^1+\mathbf{n}^2$ and $\mathbf{n}=\mathbf{n}^2$. Since $\partial_\Lambda(\mathbf{n}^1+\mathbf{n}^2) = \partial_\Lambda\mathbf{n}^1 \triangle \partial_\Lambda\mathbf{n}^2$, $\mathbf{n}\le\mathbf{m}$ and

$$\mathrm{w}(\mathbf{n}^1)\mathrm{w}(\mathbf{n}^2) = \binom{\mathbf{n}^1+\mathbf{n}^2}{\mathbf{n}^2}\mathrm{w}(\mathbf{n}^1+\mathbf{n}^2) = \binom{\mathbf{m}}{\mathbf{n}}\mathrm{w}(\mathbf{m}),$$

we can write

$$\sum_{\substack{\partial_\Lambda\mathbf{n}^1=A\\ \partial_\Lambda\mathbf{n}^2=\{i\}\\ \mathbf{n}^1+\mathbf{n}^2\in\mathscr{I}}} \mathrm{w}(\mathbf{n}^1)\mathrm{w}(\mathbf{n}^2) = \sum_{\substack{\partial_\Lambda\mathbf{m}=A\triangle\{i\}\\ \mathbf{m}\in\mathscr{I}}} \mathrm{w}(\mathbf{m}) \sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=\{i\}}} \binom{\mathbf{m}}{\mathbf{n}}. \tag{3.79}$$

The first observation is that $i \overset{\mathbf{m}}{\not\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda \implies i \overset{\mathbf{n}}{\not\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda$, since $\mathbf{n}\le\mathbf{m}$. Consequently,

$$\sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=\{i\}}} \binom{\mathbf{m}}{\mathbf{n}} = 0, \quad \text{when } i \overset{\mathbf{m}}{\not\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda, \tag{3.80}$$

since $i \overset{\mathbf{n}}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda$ whenever $\partial_\Lambda\mathbf{n}=\{i\}$. Let us therefore assume that $i \overset{\mathbf{m}}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda$, which allows us to use the following lemma, which will be proven below.

Lemma 3.56. *Let* $\mathbf{m}$ *be a current configuration in* $\Lambda \Subset \mathbb{Z}^d$ *and* $C,D\subset\Lambda$. *If there exists a current configuration* $\mathbf{k}$ *such that* $\mathbf{k}\le\mathbf{m}$ *and* $\partial_\Lambda\mathbf{k}=C$, *then*

$$\sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=D}} \binom{\mathbf{m}}{\mathbf{n}} = \sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=C\triangle D}} \binom{\mathbf{m}}{\mathbf{n}}. \tag{3.81}$$

An application of this lemma with $C=D=\{i\}$ yields

$$\sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=\{i\}}} \binom{\mathbf{m}}{\mathbf{n}} = \sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=\varnothing}} \binom{\mathbf{m}}{\mathbf{n}}, \quad \text{when } i \overset{\mathbf{m}}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda. \tag{3.82}$$

Using (3.80) and (3.82) in (3.79), and returning to the variables $\mathbf{n}^1=\mathbf{m}-\mathbf{n}$ and $\mathbf{n}^2=\mathbf{n}$, we get

$$\begin{aligned}\sum_{\substack{\partial_\Lambda\mathbf{n}^1=A\\ \partial_\Lambda\mathbf{n}^2=\{i\}\\ \mathbf{n}^1+\mathbf{n}^2\in\mathscr{I}}} \mathrm{w}(\mathbf{n}^1)\mathrm{w}(\mathbf{n}^2) &= \sum_{\substack{\partial_\Lambda\mathbf{m}=A\triangle\{i\}\\ \mathbf{m}\in\mathscr{I}\\ i\overset{\mathbf{m}}{\longleftrightarrow}\partial^{\mathrm{ex}}\Lambda}} \mathrm{w}(\mathbf{m}) \sum_{\substack{\mathbf{n}\le\mathbf{m}\\ \partial_\Lambda\mathbf{n}=\varnothing}} \binom{\mathbf{m}}{\mathbf{n}}\\ &= \sum_{\substack{\partial_\Lambda\mathbf{n}^1=A\triangle\{i\}\\ \partial_\Lambda\mathbf{n}^2=\varnothing\\ \mathbf{n}^1+\mathbf{n}^2\in\mathscr{I}}} \mathrm{w}(\mathbf{n}^1)\mathrm{w}(\mathbf{n}^2)\, \mathbf{1}_{\{i\overset{\mathbf{n}^1+\mathbf{n}^2}{\longleftrightarrow}\partial^{\mathrm{ex}}\Lambda\}},\end{aligned}$$

and the proof is complete. □

Proof of Lemma 3.56. Let us associate to the configuration $\mathbf{m}$ the graph $G_{\mathbf{m}}$ with vertices $\Lambda \cup \partial^{\mathrm{ex}}\Lambda$ and with m_e edges between the endpoints of each edge $e \in \mathscr{E}^{\mathrm{b}}_{\Lambda}$. By assumption, $G_{\mathbf{m}}$ possesses a subgraph $G_{\mathbf{k}}$ with $\partial_\Lambda G_{\mathbf{k}} = C$, where $\partial_\Lambda G_{\mathbf{k}}$ is the set of vertices of Λ belonging to an odd number of edges.

The left-hand side of (3.81) is equal to the number of subgraphs G of $G_{\mathbf{m}}$ satisfying $\partial_\Lambda G = D$, while the right-hand side counts the number of subgraphs G of $G_{\mathbf{m}}$ satisfying $\partial_\Lambda G = C \triangle D$. But the application $G \mapsto G \triangle G_{\mathbf{k}}$ defines a bijection between these two families of graphs, since $\partial_\Lambda(G \triangle G_{\mathbf{k}}) = \partial_\Lambda G \triangle \partial_\Lambda G_{\mathbf{k}}$ and $(G \triangle G_{\mathbf{k}}) \triangle G_{\mathbf{k}} = G$. □

As one simple application of the Switching Lemma, let us derive a probabilistic representation for the truncated 2-point function.

Lemma 3.57. *For all distinct $i, j \in \Lambda \Subset \mathbb{Z}^d$,*

$$\langle \sigma_i ; \sigma_j \rangle^+_{\Lambda;\beta,0} = \frac{\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda \mathbf{n}^1 = \{i,j\}, \partial_\Lambda \mathbf{n}^2 = \varnothing, i \overset{\mathbf{n}^1+\mathbf{n}^2}{\not\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda)}{\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda \mathbf{n}^1 = \varnothing, \partial_\Lambda \mathbf{n}^2 = \varnothing)} . \tag{3.83}$$

Proof. Using the representation of Exercise 3.38,

$$\begin{aligned}\langle \sigma_i ; \sigma_j \rangle^+_{\Lambda;\beta,0} &= \frac{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda \mathbf{n} = \{i,j\})}{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda \mathbf{n} = \varnothing)} - \frac{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda \mathbf{n} = \{i\})}{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda \mathbf{n} = \varnothing)} \frac{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda \mathbf{n} = \{j\})}{\mathbb{P}^+_{\Lambda;\beta}(\partial_\Lambda \mathbf{n} = \varnothing)} \\ &= \frac{\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda \mathbf{n}^1 = \{i,j\}, \partial_\Lambda \mathbf{n}^2 = \varnothing) - \mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda \mathbf{n}^1 = \{i\}, \partial_\Lambda \mathbf{n}^2 = \{j\})}{\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda \mathbf{n}^1 = \varnothing, \partial_\Lambda \mathbf{n}^2 = \varnothing)} .\end{aligned}$$

Since the Switching Lemma implies that

$$\mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda \mathbf{n}^1 = \{i\}, \partial_\Lambda \mathbf{n}^2 = \{j\}) = \mathbb{P}^{+(2)}_{\Lambda;\beta}(\partial_\Lambda \mathbf{n}^1 = \{i,j\}, \partial_\Lambda \mathbf{n}^2 = \varnothing, i \overset{\mathbf{n}^1+\mathbf{n}^2}{\longleftrightarrow} \partial^{\mathrm{ex}}\Lambda),$$

we can cancel terms in the numerator and the conclusion follows. □

Observe that (3.83) implies that $\langle \sigma_i ; \sigma_j \rangle^+_{\Lambda;\beta,0} \ge 0$, which is a particular instance of the GKS (or FKG) inequalities. However, having such a probabilistic representation also opens up the possibility of proving nontrivial lower and upper bounds.

Among the numerous fundamental applications of the random-current representation, let us mention the proof that $m^*(\beta_c(d)) = 0$ in all dimensions $d \ge 2$ [3, 7, 8], the proof that, for all $\beta < \beta_c(d)$ and all $d \ge 1$, there exists $c = c(\beta, d) > 0$ such that $\langle \sigma_0 \sigma_i \rangle^+_{\beta,0} \le e^{-c\|i\|_2}$ [5], the fact that $\langle \sigma_0 \sigma_i \rangle_{\beta_c(d),0} \simeq c_d \|i\|_2^{2-d}$ in all large enough dimensions [292] and the determination of the sign of all Ursell functions in [306]. Additional information can be found in the references given above.

3.10.7 Non-translation-invariant Gibbs states and interfaces.

In this subsection, we briefly discuss the existence or absence of non-translation-invariant Gibbs states describing coexistence of phases. The first proof of the existence of non-translation-invariant Gibbs states in the Ising model on $\mathbb{Z}^d$, $d \ge 3$, at sufficiently low temperatures, is due to Dobrushin [81]; the much simpler argument we provide below is due to van Beijeren [338].

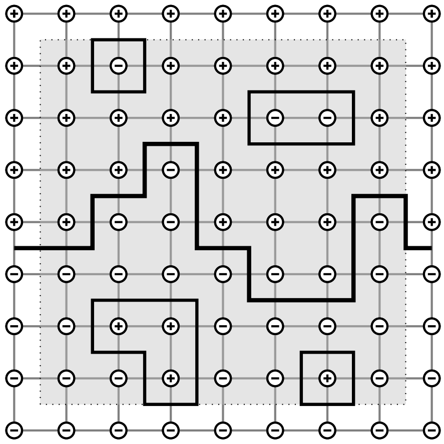


Figure 3.14: In $d = 2$, with Dobrushin boundary condition, a configuration always has a unique open contour (called interface in the text, the thickest line on the figure) connecting the two vertical sides of the box.

We require the parameters of the model to be such that the system is in the non-uniqueness regime. So, for the rest of the section, we always assume that $d \geq 2$, $h = 0$ and $\beta > \beta_c(d)$.

A natural way to try to induce spatial coexistence of the + and − phases in a system is to use non-homogeneous boundary conditions. Let us therefore consider the **Dobrushin boundary condition** η^{Dob}, defined by (see Figure 3.14)

$$\eta_i^{\text{Dob}} \stackrel{\text{def}}{=} \begin{cases} +1 & \text{if } i = (i_1, \dots, i_d) \text{ with } i_d \geq 0, \\ -1 & \text{otherwise.} \end{cases}$$

Let us then define the sequence of boxes to be used for the rest of the section, more suited to the use of the Dobrushin boundary condition,

$$\Lambda^d(n) \stackrel{\text{def}}{=} \{ i \in \mathbb{Z}^d : -n \leq i_j \leq n \text{ if } 1 \leq j < d,\ -n \leq i_d \leq n-1 \},$$

If $i = (i_1, i_2, \dots, i_{d-1}, i_d) \in \mathbb{Z}^d$, we denote by $\bar{i} = (i_1, i_2, \dots, i_{d-1}, -1 - i_d) \in \mathbb{Z}^d$ its reflection through the plane $\{x \in \mathbb{R}^d : x_d = -\frac{1}{2}\}$.

The non-homogeneity of the Dobrushin boundary condition can be shown to have a significant effect in higher dimensions:

Theorem 3.58. *Assume $d \geq 3$. Then, for all $\beta > \beta_c(d-1)$, there exists a sequence of integers $n_k \uparrow \infty$ along which*

$$\langle \cdot \rangle_{\beta,0}^{\text{Dob}} \stackrel{\text{def}}{=} \lim_{k \to \infty} \langle \cdot \rangle_{\Lambda^d(n_k);\beta,0}^{\text{Dob}}$$

is a well-defined Gibbs state that satisfies

$$\langle \sigma_0 \rangle_{\beta,0}^{\text{Dob}} > 0 > \langle \sigma_{\bar{0}} \rangle_{\beta,0}^{\text{Dob}}.$$

In particular, $\langle \cdot \rangle_{\beta,0}^{\text{Dob}}$ is not invariant under vertical translations.

The states constructed in the previous theorem are usually called **Dobrushin states**. The proof of this result relies on the following key inequality:

Proposition 3.59. *Let $d \geq 2$. Then, for all $i \in \mathsf{B}^d(n)$ such that $i_d = 0$,*

$$\langle \sigma_i \rangle^{\mathrm{Dob}}_{\mathsf{B}^d(n);\beta,0} \geq \langle \sigma_i \rangle^{+}_{\mathsf{B}^{d-1}(n);\beta,0}, \tag{3.84}$$

where the expectation in the right-hand side is for the Ising model in $\mathbb{Z}^{d-1}$.

Proof of Proposition 3.59. We use an argument due to van Beijeren [338]. To simplify notations, we stick to the case $d = 3$, but the argument can be adapted in a straightforward way to higher dimensions. To show that

$$\langle \sigma_0 \rangle^{\mathrm{Dob}}_{\mathsf{B}^3(n);\beta,0} \geq \langle \sigma_0 \rangle^{+}_{\mathsf{B}^2(n);\beta,0}, \tag{3.85}$$

the idea is to couple the two-dimensional Ising model in the box $\mathsf{B}^2(n)$ with the layer $\mathsf{B}^{3,0}(n) \stackrel{\text{def}}{=} \{i \in \mathsf{B}^3(n) : i_3 = 0\}$ of the three-dimensional model. It will be convenient to distinguish the spins of the three-dimensional model and those of the two-dimensional one. We thus continue to denote by σ_i the former, but we write τ_i for the latter. We then introduce new random variables. For all $i \in \mathsf{B}^{3,+}(n) \stackrel{\text{def}}{=} \{i \in \mathsf{B}^3(n) : i_3 > 0\}$, we set

$$s_i \stackrel{\text{def}}{=} \tfrac{1}{2}(\sigma_i + \sigma_{\underline{i}}), \qquad t_i \stackrel{\text{def}}{=} \tfrac{1}{2}(\sigma_i - \sigma_{\underline{i}}),$$

where , for $i = (i_1, i_2, i_3)$, we have set $\underline{i} \stackrel{\text{def}}{=} (i_1, i_2, -i_3)$. Moreover, for all $i \in \mathsf{B}^{3,0}(n)$, we set

$$s_i \stackrel{\text{def}}{=} \tfrac{1}{2}(\sigma_i + \tau_i), \qquad t_i \stackrel{\text{def}}{=} \tfrac{1}{2}(\sigma_i - \tau_i).$$

These random variables are $\{-1, 0, 1\}$-valued and satisfy the constraint

$$s_i = 0 \Leftrightarrow t_i \neq 0, \qquad \forall i \in \mathsf{B}^{3,+}(n) \cup \mathsf{B}^{3,0}(n). \tag{3.86}$$

Observe now that (3.85) is equivalent to

$$\langle t_0 \rangle \geq 0, \tag{3.87}$$

where the expectation is with respect to $\mu^{\mathrm{Dob}}_{\mathsf{B}^3(n);\beta,0} \otimes \mu^{+}_{\mathsf{B}^2(n);\beta,0}$. The conclusion thus follows from Exercise 3.39 below. □

(3.87) is actually a particular instance of a set of GKS-type inequalities, originally studied by Percus.

Exercise 3.39. *Prove* (3.87). Hint: *Expand the numerator of $\langle t_0 \rangle$ according to the realization of $A = \{i \in \mathsf{B}^{3,+}(n) \cup \mathsf{B}^{3,0}(n) : s_i = 0\}$. Observe that, once A is fixed, there remains exactly one nontrivial $\{-1,1\}$-valued variable at each vertex. Verify that you can then apply the usual GKS inequalities to show that each term of the sum is nonnegative (you will have to check that the resulting Hamiltonian has the proper form).*

Proof of Theorem 3.58. The construction of $\langle \cdot \rangle^{\mathrm{Dob}}_{\beta,0}$ along some subsequence $\Lambda^d(n_k)$ can be done as in Exercise 3.8. Observe that, by symmetry,

$$\langle \sigma_0 \rangle^{\mathrm{Dob}}_{\Lambda^d(n_k);\beta,0} = -\langle \sigma_{\bar{0}} \rangle^{\mathrm{Dob}}_{\Lambda^d(n_k);\beta,0}, \tag{3.88}$$

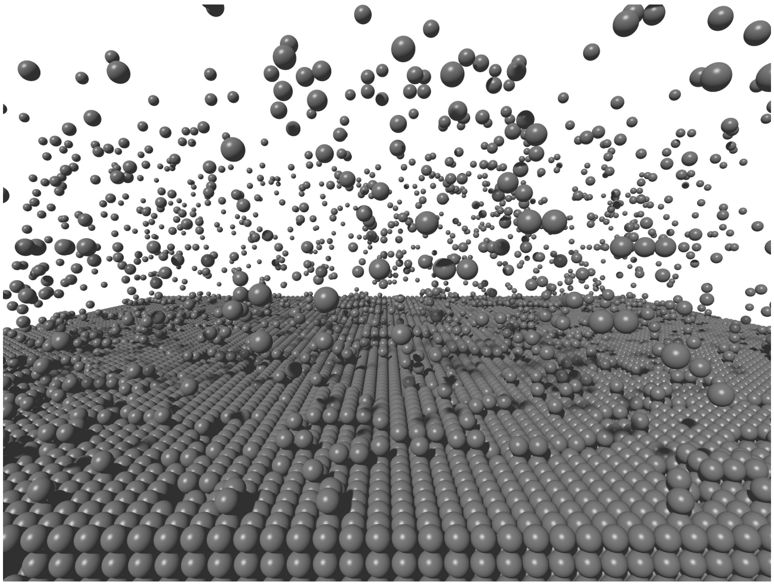


Figure 3.15: A typical configuration of the low-temperature three-dimensional Ising model with Dobrushin boundary condition. For convenience, in this picture, − spins are represented by balls and + spins by empty space. The interface is a perfect plane with only local defects.

which gives, after $k \to \infty$,

$$\langle \sigma_0 \rangle^{\mathrm{Dob}}_{\beta,0} = -\langle \sigma_{\bar{0}} \rangle^{\mathrm{Dob}}_{\beta,0}. \tag{3.89}$$

Observe that, by the FKG inequality, applying a magnetic field $h \uparrow \infty$ on the spins living in $\mathsf{B}^d(n_k) \setminus \Lambda^d(n_k)$ yields

$$\langle \sigma_0 \rangle^{\mathrm{Dob}}_{\Lambda^d(n_k);\beta,0} \geq \langle \sigma_0 \rangle^{\mathrm{Dob}}_{\mathsf{B}^d(n_k);\beta,0}.$$

Using (3.84), we deduce that

$$\langle \sigma_0 \rangle^{\mathrm{Dob}}_{\Lambda^d(n_k);\beta,0} \geq \langle \sigma_0 \rangle^{+}_{\mathsf{B}^{d-1}(n_k);\beta,0}.$$

The limit $k \to \infty$ of the right-hand side converges to the spontaneous magnetization of the $(d-1)$-dimensional Ising model, which is positive when $\beta > \beta_c(d-1)$. The claim thus follows from (3.89). □

The interface. Whether non-translation-invariant infinite-volume Gibbs states exist in $d \geq 3$ is in fact closely related to the behavior of the *macroscopic interface* induced by the Dobrushin boundary condition.

Let $\omega \in \Omega^{\mathrm{Dob}}_{\Lambda^d(n)}$ and consider the set

$$\mathscr{B}(\omega) \stackrel{\mathrm{def}}{=} \bigcup_{\substack{\{i,j\} \in \mathscr{E}_{\mathbb{Z}^d} \\ \omega_i \neq \omega_j}} \pi_{ij},$$

where each $\pi_{ij} \stackrel{\mathrm{def}}{=} \mathscr{S}_i \cap \mathscr{S}_j$ (remember (3.31)) is called a **plaquette**. By construction, $\mathscr{B}$ contains a unique infinite connected component (coinciding with the plane $\{x \in$

$\mathbb{R}^d : x_d = -\frac{1}{2}\}$ everywhere outside $\Lambda^d(n)$). We call this component the **interface** and denote it by $\Gamma = \Gamma(\omega)$.

It turns out that, in $d \geq 3$, Γ is *rigid* at low temperature: in typical configurations, Γ coincides with $\{x_d = -\frac{1}{2}\}$ apart from local defects; see Figure 3.15. This can be quantified very precisely using cluster expansion techniques, as was done in Dobrushin's original work [81]. The much simpler description given below provides substantially less information, but still allows to prove localization of Γ in a weaker sense.

Theorem 3.60. *Assume that $d \geq 3$. There exists $c'(\beta) > 0$ satisfying $\lim_{\beta\to\infty} c'(\beta) = 0$ such that, uniformly in n and in $i \in \{j \in \Lambda^d(n) : j_d = 0\}$,*

$$\mu^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0}(\Gamma \supset \pi_{i\bar{i}}) \geq 1 - c'(\beta).$$

Proof of Theorem 3.60. We first decompose

$$\langle \sigma_i \sigma_{\bar{i}} \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0} = \langle \sigma_i \sigma_{\bar{i}} \mathbf{1}_{\{\Gamma \supset \pi_{i\bar{i}}\}} \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0} + \langle \sigma_i \sigma_{\bar{i}} \mathbf{1}_{\{\Gamma \not\supset \pi_{i\bar{i}}\}} \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0}. \tag{3.90}$$

On the one hand, $\sigma_i \sigma_{\bar{i}} = -1$ whenever $\Gamma \supset \pi_{i\bar{i}}$. On the other hand, when $\Gamma \not\supset \pi_{i\bar{i}}$, i and $\bar{i}$ belong to the same (random) component of $\Lambda^d(n) \setminus \Gamma$, with constant (either + or −) boundary condition. More precisely, to a fixed interface Γ we associate a partition of $\Lambda^d(n)$ into connected regions $D_1, \dots, D_k$. The Dobrushin boundary condition, together with Γ, induces a well-defined constant boundary condition $\#_i$ on each region D_i, either + or −. $\Gamma \not\supset \pi_{i\bar{i}}$ means that the edge $\{i, \bar{i}\}$ is contained inside one of these components, say D_*. We can therefore write

$$\langle \sigma_i \sigma_{\bar{i}} \mathbf{1}_{\{\Gamma \not\supset \pi_{i\bar{i}}\}} \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0} = \sum_{\Gamma \not\supset \pi_{i\bar{i}}} \langle \sigma_i \sigma_{\bar{i}} \rangle^{\#_*}_{D_*;\beta,0} \, \mu^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0}(\Gamma(\omega) = \Gamma). \tag{3.91}$$

Assume that $\#_* = +$. Then the GKS inequalities (see Exercise 3.12) imply that

$$\langle \sigma_i \sigma_{\bar{i}} \rangle^+_{D_*;\beta,0} \geq \langle \sigma_i \sigma_{\bar{i}} \rangle^+_{\Lambda^d(n);\beta,0} \geq \langle \sigma_i \sigma_{\bar{i}} \rangle^+_{\beta,0}.$$

When $\#_* = -$, the same holds since, by symmetry, $\langle \sigma_i \sigma_{\bar{i}} \rangle^+_{D_*;\beta,0} = \langle \sigma_i \sigma_{\bar{i}} \rangle^-_{D_*;\beta,0}$ and $\langle \sigma_i \sigma_{\bar{i}} \rangle^+_{\beta,0} = \langle \sigma_i \sigma_{\bar{i}} \rangle^-_{\beta,0}$. Thus,

$$\langle \sigma_i \sigma_{\bar{i}} \mathbf{1}_{\{\Gamma \not\supset \pi_{i\bar{i}}\}} \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0} \geq \langle \sigma_i \sigma_{\bar{i}} \rangle^+_{\beta,0} \, \mu^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0}(\Gamma \not\supset \pi_{i\bar{i}}).$$

Collecting the above and rearranging the terms, we get

$$\mu^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0}(\Gamma \supset \pi_{i\bar{i}}) \geq 1 - \frac{1 + \langle \sigma_i \sigma_{\bar{i}} \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0}}{1 + \langle \sigma_i \sigma_{\bar{i}} \rangle^+_{\beta,0}}.$$

Let us consider the numerator in the right-hand side. Using Jensen's inequality, we can write

$$\langle \sigma_i \sigma_{\bar{i}} \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0} = 1 - \tfrac{1}{2} \big\langle (\sigma_i - \sigma_{\bar{i}})^2 \big\rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0} \leq 1 - \tfrac{1}{2} \big(\langle \sigma_i - \sigma_{\bar{i}} \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0} \big)^2.$$

But $\langle \sigma_{\bar{i}} \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0} = -\langle \sigma_i \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0}$ and so, by Proposition 3.59,

$$\langle \sigma_i - \sigma_{\bar{i}} \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0} = 2 \langle \sigma_i \rangle^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0} \geq 2 \langle \sigma_0 \rangle^+_{\beta,0;d-1},$$

from which we conclude that

$$\mu^{\mathrm{Dob}}_{\Lambda^d(n);\beta,0}(\Gamma \supset \pi_{i\bar{i}}) \geq 1 - 2\frac{1 - \left(\langle\sigma_0\rangle^+_{\beta,0;d-1}\right)^2}{1 + \langle\sigma_i\sigma_{\bar{i}}\rangle^+_{\beta,0}} \geq 1 - 2\frac{1 - \left(\langle\sigma_0\rangle^+_{\beta,0;d-1}\right)^2}{1 + \left(\langle\sigma_0\rangle^+_{\beta,0}\right)^2}.$$

This lower bound is uniform in n and i and converges to 1 as $\beta \to \infty$. □

Of course, Theorem 3.58 only shows the existence of non-translation-invariant Gibbs states when $\beta > \beta_c(d-1)$, and one might wonder what happens for values of β in the remaining interval $(\beta_c(d), \beta_c(d-1)]$. It turns out that this problem is still open. The conjectured behavior, however, is as follows: [5]

- When $d = 3$, there should exist a value $\beta_R \in (\beta_c(3), \beta_c(2)]$ such that the existence of Gibbs states which are not translation invariant holds for all $\beta > \beta_R$, but not for $\beta < \beta_R$. At β_R, the system is said to undergo a **roughening transition**. At this transition the interface is supposed to lose its rigidity and to start having unbounded fluctuations. [6]
- When $d \geq 4$, Dobrushin's non-translation-invariant Gibbs states are believed to exist (with a rigid interface) for all $\beta > \beta_c(d)$.

Two-dimensional model. The behavior of the interface in two dimensions is very different and, from a mathematical point of view, a rather detailed and complete picture is available.

Consider again a configuration $\omega \in \Omega^{\mathrm{Dob}}_{\Lambda_n}$ and, in particular, the associated interface Γ. Let us denote by ω_Γ the configuration in $\Omega^{\mathrm{Dob}}_{\Lambda_n}$ for which $\mathscr{B}(\omega_\Gamma) = \{\Gamma\}$. We can then define the upper and lower "envelopes" $\Gamma^\pm : \mathbb{Z} \to \mathbb{Z}$ of Γ by

$$\Gamma^+(i) \stackrel{\mathrm{def}}{=} \max\{j \in \mathbb{Z} : \sigma_{(i,j)}(\omega_\Gamma) = -1\} + 1,$$
$$\Gamma^-(i) \stackrel{\mathrm{def}}{=} \min\{j \in \mathbb{Z} : \sigma_{(i,j)}(\omega_\Gamma) = +1\} - 1.$$

Note that $\Gamma^+(i) > \Gamma^-(i)$ for all $i \in \mathbb{Z}$. One can show [60] that, with probability close to 1, Γ^- and Γ^+ remain very close to each other: there exists $K = K(\beta) < \infty$ such that, with probability tending to 1 as $n \to \infty$,

$$\max_{i\in\mathbb{Z}} |\Gamma^+(i) - \Gamma^-(i)| \leq K \log n. \tag{3.92}$$

Let us now introduce the diffusively-rescaled profiles $\hat{\Gamma}^\pm : [-1,1] \to \mathbb{R}$. Given $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, let us write $\lfloor y \rfloor \stackrel{\mathrm{def}}{=} (\lfloor y_1 \rfloor, \dots, \lfloor y_d \rfloor)$. We then set, for any $x \in [-1,1]$,

$$\hat{\Gamma}^+(x) = \frac{1}{\sqrt{n}}\Gamma^+(\lfloor nx \rfloor),$$

and similarly for Γ^-. Observe that, thanks to (3.92), we know that

$$\lim_{n\to\infty} \mu^{\mathrm{Dob}}_{\Lambda_n}\Big(\sup_{x\in[-1,1]} |\hat{\Gamma}^+(x) - \hat{\Gamma}^-(x)| \leq \epsilon\Big) = 1, \text{ for all } \epsilon > 0.$$

Since the interface Γ is squeezed between Γ^+ and Γ^-, studying the limiting behavior of $\hat{\Gamma}^+$ suffices to understand the asymptotic behavior of the interface under diffusive scaling. This is the content of the next theorem, first proved by Higuchi [161] for large enough values of β and then extended to all $\beta > \beta_c(2)$ by Greenberg and Ioffe [141].

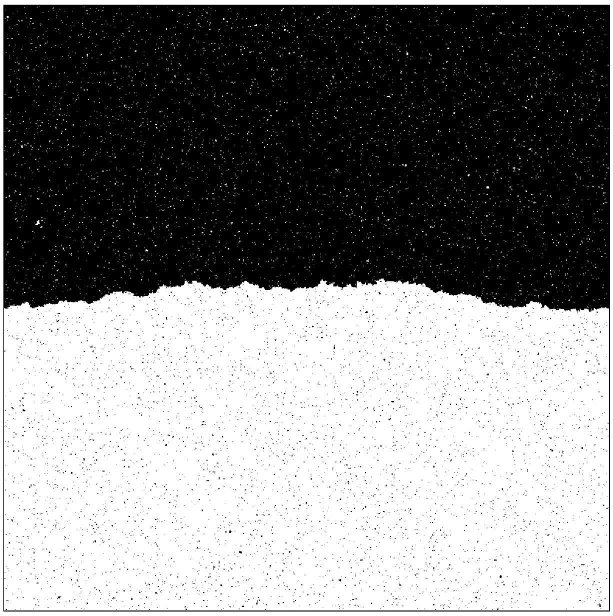


Figure 3.16: A typical configuration of the low-temperature two-dimensional Ising model with Dobrushin boundary condition. Once properly rescaled, the interface between the two phases converges weakly to a Brownian Bridge process.

Theorem 3.61. *For all $\beta > \beta_c(2)$, there exists $\kappa_\beta \in (0,\infty)$ such that $\hat{\Gamma}^+$ converges weakly to a Brownian bridge on $[-1,1]$ with diffusivity constant κ_β.*

(The Brownian bridge is a Brownian motion $(B_t)_{t\in[-1,1]}$ starting at 0 at $t=-1$ and conditioned to be at 0 at $t=+1$; see [251].) It is also possible [141] to express the diffusivity constant κ_β in terms of the physically relevant quantity, the surface tension, but this is beyond the scope of this book.

Theorem 3.61 shows that, in contrast to what happens in higher dimensions, the interface of the two-dimensional Ising model is never rigid (except in the trivial case $\beta = +\infty$); see Figure 3.16. Moreover, in a finite box Λ_n, Γ undergoes vertical fluctuations of order $\sqrt{n}$. A consequence of this delocalization of the interface is the following: when n becomes very large, the behavior of the system near the center of the box Λ_n will be typical of either the + phase (if Γ has wandered far away below the origin) or the − phase (if Γ has wandered far away above the origin), and the probability of each of these two alternatives converges to $\frac{1}{2}$ as $n\to\infty$. In particular, in this case, the infinite-volume Gibbs state resulting from Dobrushin boundary condition is translation invariant and given by $\frac{1}{2}\mu^+_{\beta,0} + \frac{1}{2}\mu^-_{\beta,0}$. More details and far-reaching generalizations are discussed in Section 3.10.8.

3.10.8 Gibbs states and local behavior in large finite systems

When introducing the notion of Gibbs state in Section 3.4, we motivated the definition by saying that the latter should lead to an interpretation of Gibbs states as providing approximate descriptions of all possible local behaviors in large finite systems, the quality of this approximation improving with the distance to the system's boundary. It turns out that, in the two-dimensional Ising model, this heuristic discussion can be made precise and rigorous.

Let us consider an arbitrary finite subset $\Lambda \Subset \mathbb{Z}^d$ and an arbitrary boundary condition $\eta \in \Omega$. We are interested in describing the local behavior of the Gibbs distribution $\mu^{\eta}_{\Lambda;\beta,0}$ in the vicinity of a point $i \in \Lambda$. Since Λ and η are arbitrary, there is no loss of generality in assuming that $i = 0$.

The case of pure boundary conditions. Let us first consider the simpler case of constant boundary conditions, which we will assume to be + for the sake of concreteness. We know from the definition of $\langle\cdot\rangle^+_{\beta,0}$ that, for any local function f,

$$\langle f\rangle^+_{\Lambda;\beta,0} \to \langle f\rangle^+_{\beta,0} \quad \text{as } \Lambda \uparrow \mathbb{Z}^d.$$

We will now state a result, first proved by Bricmont, Lebowitz and Pfister [49], that says that $\langle\cdot\rangle^+_{\beta,0}$ actually provides an approximation for the finite-volume expectation $\langle\cdot\rangle^+_{\Lambda;\beta,0}$ with an error exponentially small in the distance from the support of f to the boundary of the box. Let

$$R \overset{\text{def}}{=} \max\{n : \mathsf{B}(n) \subset \Lambda\}$$

denote the distance from the origin to boundary of Λ and let $r \overset{\text{def}}{=} \lfloor R/2 \rfloor$.

Theorem 3.62 (Exponential relaxation)**.** *Assume that $\beta > \beta_c(2)$. There exists $c_1 = c_1(\beta) > 0$ such that the following holds. Let $\Lambda \Subset \mathbb{Z}^2$. Then, uniformly in all functions f with* $\mathrm{supp}(f) \subset \mathsf{B}(r)$,

$$\big|\langle f\rangle^+_{\Lambda;\beta,0} - \langle f\rangle^+_{\beta,0}\big| \le \tfrac{1}{c_1}\|f\|_\infty e^{-c_1 R}.$$

The same holds for the − boundary condition.

This fully vindicates the statement that the Gibbs state $\langle\cdot\rangle^+_{\beta,0}$ (resp. $\langle\cdot\rangle^-_{\beta,0}$) provides an accurate description of the local behavior of any finite-volume Gibbs distribution with + (resp. −) boundary condition, in regions of size proportional to the distance to the boundary of the system.

The case of general boundary conditions. Let us now turn to the case of a Gibbs distribution with an arbitrary boundary condition η, which is much more delicate. For $\Lambda \Subset \mathbb{Z}^d$, take R as before, but this time define r as follows: fix some small $\epsilon \in (0, 1/2)$ and set

$$r \overset{\text{def}}{=} \lfloor R^{1/2-\epsilon} \rfloor. \tag{3.93}$$

We call **circuit** a set of distinct vertices $(t_0, t_1, \dots, t_k)$ of $\mathbb{Z}^2$ with the property that $\|t_m - t_{m-1}\|_\infty = 1$, for all $1 \le m \le k$, and $\|t_k - t_0\|_\infty = 1$. Let $\mathscr{C}_\epsilon$ be the event that there is a circuit surrounding $\mathsf{B}(2r)$ in $\Lambda \cup \partial^{\text{ex}}\Lambda$ and along which the spins take a constant value. We decompose $\mathscr{C}_\epsilon = \mathscr{C}^+_\epsilon \cup \mathscr{C}^-_\epsilon$ according to the sign of the spins along the outermost such circuit. The main observation is that the event $\mathscr{C}_\epsilon$ is typical when $\beta > \beta_c(2)$, a fact first proved by Coquille and Velenik [73].

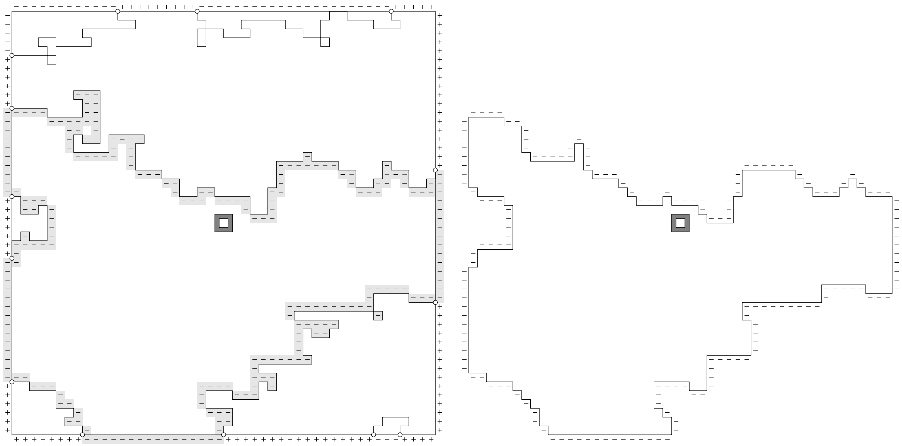


Figure 3.17: A (square) box Λ with a non-constant boundary condition. The boundary condition induces open Peierls contours inside the system. With probability close to 1, none of them intersect the box $\mathsf{B}(2r)$ located in the middle (represented by the dark square). Left: A realization of the open Peierls contours. The event $\mathscr{C}_\epsilon^-$ occurs. The relevant $-$ spins, the value of which is forced by the realization of the open contours, are indicated (and shaded). Right: The induced random box with $-$ boundary condition. The box $\mathsf{B}(r)$, represented by the small white square in the middle, is located at a distance at least r from the boundary of this box.

Theorem 3.63. *Assume that $\beta > \beta_c(2)$. For all $\epsilon > 0$, there exists $c_2 = c_2(\beta, \epsilon) > 0$ such that*

$$\mu^{\eta}_{\Lambda;\beta,0}(\mathscr{C}_\epsilon) \geq 1 - c_2 R^{-\epsilon},$$

uniformly in $\Lambda \Subset \mathbb{Z}^2$ and $\eta \in \Omega$.

Therefore, neglecting an event of probability at most $c_2 R^{-\epsilon}$, we can assume that one of the events $\mathscr{C}_\epsilon^+$ or $\mathscr{C}_\epsilon^-$ occurs. For definiteness, let us consider the latter case and let us denote by π the corresponding outermost circuit. Observe now that, conditionally on $\mathscr{C}_\epsilon^-$ and π, any function f with $\mathrm{supp}(f) \subset \mathsf{B}(r)$ finds itself in a box (delimited by π) with $-$ boundary condition (see Figure 3.17). Moreover, its support is at a distance at least r from the boundary of this box. It thus follows from Theorem 3.62 that its (conditional) expectation is closely approximated by $\langle f \rangle^-_{\beta,0}$. This leads [73] to the following generalization of Theorem 3.62.

Theorem 3.64. *Assume that $\beta > \beta_c(2)$. There exist constants $\alpha = \alpha(\Lambda, \eta, \beta)$ and $c_3 = c_3(\beta)$ such that, uniformly in functions f with* $\mathrm{supp}(f) \subset \mathsf{B}(r)$*, one has*

$$\Big| \langle f \rangle^{\eta}_{\Lambda;\beta,0} - \big(\alpha \langle f \rangle^+_{\beta,0} + (1-\alpha) \langle f \rangle^-_{\beta,0}\big) \Big| \leq c_3 \, \|f\|_\infty \, R^{-\epsilon}. \tag{3.94}$$

The coefficients α and $1-\alpha$ in (3.94) are given by

$$\alpha = \mu^{\eta}_{\Lambda;\beta,0}(\mathscr{C}_\epsilon^+ \,|\, \mathscr{C}_\epsilon), \qquad 1-\alpha = \mu^{\eta}_{\Lambda;\beta,0}(\mathscr{C}_\epsilon^- \,|\, \mathscr{C}_\epsilon),$$

that is, by the probabilities that the box $\mathsf{B}(R)$ (in which one measures f) finds itself deep inside a +, resp. $-$, region (conditionally on the typical event $\mathscr{C}_\epsilon$).

Again, the statement (3.94) fully vindicates the interpretation of Gibbs states as describing all possible local behaviors of any finite-volume system, with an accuracy improving with the distance to the system's boundary. This requires however, in general, that the size of the observation window be chosen small compared to the square-root of the distance to the boundary. We will explain the reason for this restriction at the end of the section.

Note that (3.94) also shows that, in the two-dimensional Ising model, the only possible local behaviors are those corresponding to the + and − phases, since the approximation is stated in terms of the two Gibbs states $\langle\cdot\rangle^+_{\beta,0}$ and $\langle\cdot\rangle^-_{\beta,0}$. In other words, looking at local properties of the system, one will see behavior typical of the + phase with probably close to α, and of the − phase with probability close to $1-\alpha$. Actually this can be made a little more precise, as we explain now.

What are the possible Gibbs states? Let us consider a sequence of boundary conditions $(\eta_n)_{n\geq 1}$ and a sequence of boxes $\Lambda_n \uparrow \mathbb{Z}^2$. We assume that the corresponding sequence of Gibbs distributions $(\mu^{\eta_n}_{\Lambda_n;\beta,0})_{n\geq 1}$ converges to some Gibbs state $\langle\cdot\rangle$. Then, applying (3.94) with $f=\sigma_0$, we conclude that

$$\lim_{n\to\infty}\Bigl|\langle\sigma_0\rangle^{\eta_n}_{\Lambda_n;\beta,0}-\bigl(\alpha_n\,\langle\sigma_0\rangle^+_{\beta,0}+(1-\alpha_n)\,\langle\sigma_0\rangle^-_{\beta,0}\bigr)\Bigr|=0\,.$$

Since, by assumption, $\lim_{n\to\infty}\langle\sigma_0\rangle^{\eta_n}_{\Lambda_n;\beta,0}=\langle\sigma_0\rangle$, this implies the existence of

$$\alpha \overset{\text{def}}{=} \lim_{n\to\infty}\alpha_n=\frac{\langle\sigma_0\rangle-\langle\sigma_0\rangle^-_{\beta,0}}{\langle\sigma_0\rangle^+_{\beta,0}-\langle\sigma_0\rangle^-_{\beta,0}}\,.$$

Applying again (3.94) to arbitrary local functions, we conclude that

$$\langle\cdot\rangle=\lim_{n\to\infty}\langle\cdot\rangle^{\eta_n}_{\Lambda_n;\beta,0}=\alpha\,\langle\cdot\rangle^+_{\beta,0}+(1-\alpha)\,\langle\cdot\rangle^-_{\beta,0}\,,$$

and thus all possible Gibbs states are convex combinations of the Gibbs states $\langle\cdot\rangle^+_{\beta,0}$ and $\langle\cdot\rangle^-_{\beta,0}$. This is the **Aizenman–Higuchi theorem**, originally derived by Aizenman and Higuchi [160] directly for infinite-volume states; see also [135] for a self-contained, somewhat simpler and more general argument.

A more general formulation of the previous derivation will be presented in Chapter 6, once we have introduced the notion of infinite-volume Gibbs measures.

As we have seen in Section 3.10.7, when $d\geq 3$ and β is large enough, there exist Gibbs states which are not translation invariant. In particular, this implies that the Aizenman–Higuchi theorem does not extend to this setting. Nevertheless, it can proved that all *translation-invariant* Gibbs states of the Ising model on $\mathbb{Z}^d$, $d\geq 3$ are convex combinations of $\langle\cdot\rangle^+_{\beta,h}$ and $\langle\cdot\rangle^-_{\beta,h}$. This result is due to Bodineau [27], who completed earlier analyses started by Gallavotti and Miracle-Solé [129] and by Lebowitz [218].

Why this constraint on the size of the observation window? In the case of pure boundary conditions, it was possible to take an observation window with a radius proportional to the distance to the boundary. We now explain why one cannot, in general, improve Theorem 3.64 to larger windows. Let us thus consider an observation window $\mathsf{B}(r)$, with now an arbitrary radius r.

The reason is to be found in the probability of observing some "pathological" behavior in our finite system. Namely, we have seen above that, typically, the event $\mathscr{C}_\epsilon$ is realized. It turns out that, for some choices of the boundary condition η, the probability of not observing $\mathscr{C}_\epsilon$ is really of order $r/\sqrt{R}$ and thus small only when $r \ll \sqrt{R}$.

A simple example consists in considering the box $\Lambda = \mathrm{B}(n)$ with the Dobrushin boundary condition η^{Dob}, as introduced in Section 3.10.7. As explained there, in that case the open Peierls contour has fluctuations of order $\sqrt{n}$ and its scaling limit is a Brownian bridge. This implies that the probability that this contour intersects $\mathrm{B}(2r)$ is indeed of order $r/\sqrt{n}$; note that, when this occurs, the event $\mathscr{C}_\epsilon$ becomes impossible.

Remark 3.65. In the uniqueness regime, quantitative estimates are easier to obtain. Consider an Ising model either at $\beta < \beta_c(2)$ and $h = 0$, or at $h \neq 0$ and arbitrary β, and let $\langle\cdot\rangle_{\beta,h}$ denote the associated (unique) Gibbs state. Then it can be shown [49, 95] that there is again exponential relaxation: there exists a constant $c_4 = c_4(\beta,h)$ such that

$$\left|\langle f\rangle^{\eta}_{\Lambda;\beta,h} - \langle f\rangle_{\beta,h}\right| \leq c\,\|f\|_\infty\, e^{-R/c_4}\,,$$

uniformly in functions f satisfying $\mathrm{supp}(f) \subset \mathrm{B}(r)$, $r = \lfloor R/2 \rfloor$. ⋄

3.10.9 Absence of analytic continuation of the pressure.

From the point of view of complex analysis, the properties of the pressure of the Ising model that we have obtained raise natural questions, that will turn out to have physical relevance, as explained in Chapter 4, in particular in the discussion of Section 4.12.3. Since we are interested in fixing the temperature and studying the analyticity properties with respect to the magnetic field, in this section, we will denote the pressure by

$$h \mapsto \psi_\beta(h)\,.$$

For the sake of concreteness, let us consider only positive fields (by the identity $\psi_\beta(-h) = \psi_\beta(h)$, everything we say here admits an equivalent for negative fields). Although the pressure was first shown to exist on the real axis, we have seen in Theorem 3.42 that it can actually be extended to the whole half-plane $H^+ = \{\mathfrak{Re}\, h > 0\}$ as an analytic function $\psi_\beta : H^+ \to \mathbb{C}$. We will also see in Section 5.7.1 how to obtain the coefficients of the expansion of $\psi_\beta(h)$ in the variable e^{-2h}, with the latter being convergent for all $h \in H^+$. Unfortunately, these results do not provide any information on the behavior of the pressure on the boundary of H^+, $\partial H^+ \stackrel{\mathrm{def}}{=} \{\mathfrak{Re}\, h = 0\}$. In function-theoretic terms, the most natural question is whether ψ_β can be *analytically continued* outside H^+. We will thus distinguish two scenarios.

Scenario 1: Analytic continuation is possible. Analytic continuation means that there exists a strictly larger domain $H' \supset H^+$ and an analytic map $\widetilde{\psi}_\beta : H' \to \mathbb{C}$, which coincides with ψ_β on H^+, as depicted in Figure 3.18. This scenario is seen, for example, in the one-dimensional Ising model: the exact solution (3.10) guarantees that ψ_β can be continued analytically through $h = 0$, at all temperatures. Of course, since it can be defined as an analytic function on the whole real line, the analytic continuation $\widetilde{\psi}_\beta$ obtained when crossing $h = 0$ is nothing but the usual pressure: for $h < 0$, $\widetilde{\psi}_\beta(h) = \psi_\beta(h)$ (see Figure 3.4).

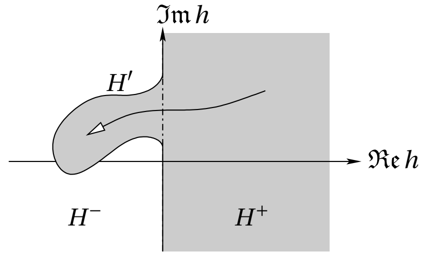


Figure 3.18: In Scenario 1, there exists an interval of the imaginary axis through which the pressure can be continued analytically.

Another example where analytic continuation is possible is provided by the Curie–Weiss model. Indeed, we have already seen in Exercise 2.4 that starting from

$$\psi_\beta^{\mathrm{CW}}(h) = \max_m \{hm - f_\beta^{\mathrm{CW}}(m)\},$$

a more explicit expression can be obtained for the pressure:

$$\psi_\beta^{\mathrm{CW}}(h) = -\frac{\beta m_\beta^{\mathrm{CW}}(h)^2}{2} + \log\cosh\big(\beta m_\beta^{\mathrm{CW}}(h) + h\big) + \log 2.$$

Although this function is not differentiable at $h = 0$ when β is large, it possesses an analytic continuation across $h = 0$. Namely, remember that $m_\beta^{\mathrm{CW}}(h)$ is the largest solution (in m) of the mean-field equation

$$\tanh(\beta m + h) = m. \tag{3.95}$$

A look at Figure 2.4 shows that the map $h \mapsto m_\beta^{\mathrm{CW}}(h)$, well-defined for $h > 0$, can obviously be continued analytically through $h = 0$, to small negative values of h, see Figure 3.19. The continuation $\tilde{m}_\beta^{\mathrm{CW}}(\cdot)$, for small $h < 0$, is still a solution of (3.95), but corresponds only to a *local* maximum of $m \mapsto hm - f_\beta^{\mathrm{CW}}(m)$, and thus does not represent the equilibrium value of the magnetization.

As a consequence, the pressure $\psi_\beta(h)$ can also be continued analytically through $h = 0$, $h \mapsto \tilde{\psi}_\beta^{\mathrm{CW}}(h)$, as depicted in Figure 3.19.

Remark 3.66. If the analytic continuation can be made to reach the negative real axis $\{h \in \mathbb{R} : h < 0\}$, as in Figure 3.19 above, then the analytically continued pressure at such (physically relevant) values of $h < 0$ can acquire an imaginary part, and some (non-rigorous) theories predict that this imaginary component should be related to the lifetime of the corresponding metastable state. See [206]. ⋄

Scenario 2: Analytic continuation is blocked by the presence of singularities. In the second scenario, there exist no analytic continuation across the imaginary axis. This happens when the singularities form a dense subset of the imaginary axis, see Figure 3.20. In such a case, $\{\mathfrak{Re}\, h = 0\}$ is called a **natural boundary** for ψ_β.

Which scenario occurs in the Ising model on $\mathbb{Z}^d$, $d \geq 2$? With the exception of the (trivial) one-dimensional case, the results concerning the possibility of analytically continuing the pressure of the Ising model across ∂H^+ are largely incomplete.

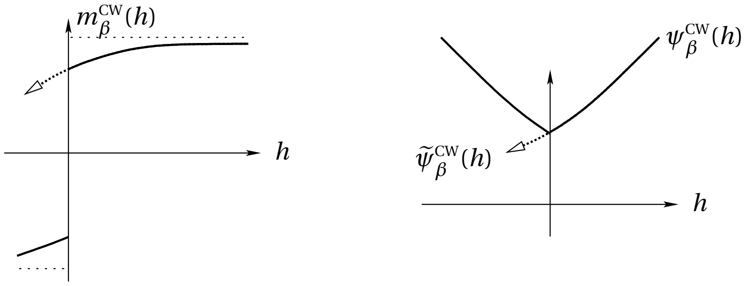


Figure 3.19: Left: the analytic continuation (dotted) of the magnetization of the Curie–Weiss model, across $h = 0$, along $h \to 0^+$. Right: the corresponding analytic continuation of the pressure. When $\beta > \beta_c$, the analytic continuation *differs* from the values of the true pressure for small $h < 0$: $\widetilde{\psi}_\beta^{\mathrm{CW}}(h) < \psi_\beta^{\mathrm{CW}}(h)$.

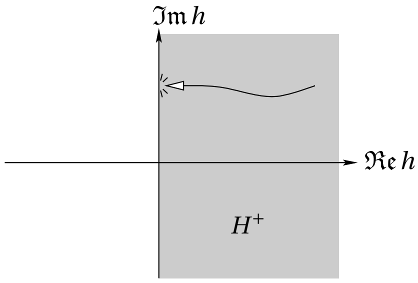


Figure 3.20: In Scenario 2, $\{\mathfrak{Re}\, h = 0\}$ is a natural boundary of the pressure: any path crossing the imaginary axis "hits" a singularity, which prevents analytic continuation.

In the supercritical regime $\beta < \beta_c(d)$, the pressure is differentiable at $h = 0$ and analytic continuation is expected to be possible, through any point of the imaginary axis. (Analyticity at $h = 0$ for the two-dimensional Ising model at any $\beta < \beta_c$ is established in [231].) For sufficiently large temperatures, a proof will be provided in Chapter 5 using the cluster expansion technique (see Exercise 5.8).

In the subcritical regime $\beta > \beta_c(d)$, the only rigorous contribution remains the study of Isakov [174], who considered the d-dimensional Ising model ($d \geq 2$) at low temperature and studied the high-order derivatives of the pressure at $h = 0$. Before stating his result, note that Theorem 3.42 allows one to use Cauchy's formula to obtain that, for all $h_0 \in H^+$,

$$\frac{\mathrm{d}^k \psi_\beta}{\mathrm{d}h^k}(h_0) = \frac{k!}{2\pi \mathrm{i}} \oint_\gamma \frac{\psi_\beta(z)}{(z - h_0)^{k+1}}\, \mathrm{d}z, \tag{3.96}$$

where γ is a smooth simple closed curve contained in H^+, surrounding h_0, oriented counterclockwise. Choosing γ as the circle of radius $|\mathfrak{Re}\, h_0|/2$ centered at h_0, we get the upper bound

$$\Big|\frac{\mathrm{d}^k \psi_\beta}{\mathrm{d}h^k}(h_0)\Big| \leq C^k k!. \tag{3.97}$$

The constant C being proportional to $1/|\mathfrak{Re}\, h_0|$, this upper bound provides no information on the behavior near the imaginary axis.

Isakov showed that, for all k, the kth one-sided derivative at $h=0$ [6], $\frac{\mathrm{d}^k\psi_\beta}{\mathrm{d}h_+^k}(0)$, exists, is finite and equals

$$\frac{\mathrm{d}^k\psi_\beta}{\mathrm{d}h_+^k}(0) = \lim_{h_0\downarrow 0}\frac{\mathrm{d}^k\psi_\beta}{\mathrm{d}h^k}(h_0)\,,$$

where the limit $h_0\downarrow 0$ is taken along the real axis. This implies that the pressure, although not differentiable at $h=0$, has right-derivatives of all orders at $h=0$. Therefore, the Taylor series for the pressure at $h=0$ exists:

$$a_0 + a_1 h + a_2 h^2 + a_3 h^3 + \dots\,, \qquad \text{where } a_k = \frac{1}{k!}\frac{\mathrm{d}^k\psi_\beta}{\mathrm{d}h_+^k}(0)\,. \tag{3.98}$$

But Isakov also obtained the following remarkable result:

Theorem 3.67. *($d\geq 2$) There exist $\beta_0<\infty$ and $0<A<B<\infty$, both depending on β, such that, for all $\beta\geq\beta_0$, as $k\to\infty$,*

$$A^k k!^{\frac{d}{d-1}} \leq \lim_{h_0\downarrow 0}\left|\frac{\mathrm{d}^k\psi_\beta}{\mathrm{d}h^k}(h_0)\right| \leq B^k k!^{\frac{d}{d-1}}\,. \tag{3.99}$$

Since $\frac{d}{d-1}>1$, (3.99) shows that the high-order derivatives at $0\in\partial H^+$ diverge much faster than inside H^+, as seen in (3.97). This implies in particular that the series (3.98) *diverges* for all $h\neq 0$, and therefore does not represent the function in a neighborhood of 0 [7]. In other words, the pressure has a singularity at $h=0$ and there exist no analytic continuation of ψ_β through the transition point. We will study this phenomenon in a simple toy model in Exercise 4.16.

Although this result has only been established at very low temperature, it is expected to hold for all $\beta>\beta_c$. Observe that, since $e^{(h+2\pi k\mathrm{i})\sigma_j}=e^{h\sigma_j}$, the pressure is periodic in the imaginary direction, with period 2π. The singularity at $h=0$ therefore implies the presence of singularities at each of the points $2\pi k\mathrm{i}\in\partial H^+$.

Isakov's result was later extended to other models (see the references at the end of Section 4.12.3). But the problem of determining whether there exists some analytic continuation *around* the singularity at $h=0$, across some interval on the imaginary axis as on Figure 3.18, is still open.

3.10.10 Metastable behavior in finite systems.

As explained in Section 3.10.9, the spontaneous magnetization of the Ising model at low temperatures cannot be analytically continued from negative values of h to positive values of h. Of course, this only applies in the thermodynamic limit, since the magnetization is an analytic function in a finite system. It is thus of interest to understand what happens, in finite systems, to the $-$ phase when h becomes positive.

To discuss this issue, let us consider the low-temperature d-dimensional Ising model in the box $\mathsf{B}(n)$ with a magnetic field h and $-$ boundary condition. When $h\leq 0$ and β is large enough, typical configurations are given by small perturbations

[6] For $k=1$, the one-sided derivative is the same as encountered earlier in the chapter: $\frac{\mathrm{d}\psi_\beta}{\mathrm{d}h^+}(0)$. For $k\geq 2$, the kth one-sided derivative is defined by induction.

of the ground state η^-: they consist in a large sea of − spins with small islands of + spins (see Exercises 3.18 and 3.19). The − boundary condition is said to be **stable** in $\mathsf{B}(n)$. The situation is more interesting when $h > 0$. To get some insight, let us consider the two configurations $\omega^-, \omega^+ \in \Omega^-_{\mathsf{B}(n)}$, in which all the spins in $\mathsf{B}(n)$ take the value −1, resp. +1. Then, $\mathscr{H}_{\mathsf{B}(n);\beta,h}(\omega^-) - \mathscr{H}_{\mathsf{B}(n);\beta,h}(\omega^+) = -2\beta|\partial^{\mathrm{ex}}\mathsf{B}(n)| + 2h|\mathsf{B}(n)|$. We thus see that ω^- and ω^+ have the same energy if and only if

$$h = \beta\frac{|\partial^{\mathrm{ex}}\mathsf{B}(n)|}{|\mathsf{B}(n)|} = \frac{2d\beta}{|\mathsf{B}(n)|^{1/d}}.$$

We would thus expect that, provided that $h > 0$ satisfies

$$h|\mathsf{B}(n)|^{1/d} < 2d\beta, \tag{3.100}$$

the − boundary condition should remain stable in $\mathsf{B}(n)$ even though there is a positive magnetic field, in the sense that typical low temperature configurations should be small perturbations of ω^-, as on the left of Figure 3.21. In contrast, when h satisfies

$$h|\mathsf{B}(n)|^{1/d} > 2d\beta, \tag{3.101}$$

one would expect the + phase to invade the box, with only a narrow layer of − phase along the boundary of $\mathsf{B}(n)$, as on the right of Figure 3.21. In this case, the − boundary condition is **unstable**.

Of course, the previous argument is very rough, taking into account only constant configurations inside $\mathsf{B}(n)$, and one should expect the above claims to be valid only for extremely low temperatures. Nevertheless, in a more careful analysis [296], Schonmann and Shlosman have showed that the above remains qualitatively true for the two-dimensional Ising model at any $\beta > \beta_c(2)$: there exists $c = c(\beta) \in (0,\infty)$ such that the − boundary condition is stable as long as $h < c|\mathsf{B}(n)|^{-1/2}$, while it becomes unstable when $h > c|\mathsf{B}(n)|^{-1/2}$. In the latter case, the macroscopic shape of the region occupied by the + phase can be characterized precisely (showing, in particular, that macroscopic regions remain occupied by the − phase near the four corners of $\mathsf{B}(n)$ as long as h is not too large). In particular, these results show that the magnetization at the center of the box satisfies, for large n and small $|h|$,

$$\langle\sigma_0\rangle^-_{\mathsf{B}(n);\beta,h} \cong \begin{cases} -m^* & \text{if } h < c|\mathsf{B}(n)|^{-1/2}, \\ +m^* & \text{if } h > c|\mathsf{B}(n)|^{-1/2}. \end{cases}$$

In this sense, the negative-h magnetization can be "continued" into the positive-h region, but only as long as $h < c|\mathsf{B}(n)|^{-1/2}$. The fact that the size of the latter interval vanishes as $n \to \infty$ explains why the above discussion does not contradict the absence of analytic continuation in the thermodynamic limit.

3.10.11 Critical phenomena.

As explained in this chapter, a first-order phase transition occurs at each point of the line $\{(\beta,h) \in \mathbb{R}_{\geq 0} \times \mathbb{R} : \beta > \beta_c(d),\, h = 0\}$, where $\beta_c(d) \in (0,\infty)$ for all $d \geq 2$. One of the manifestations of these first-order phase transitions is the discontinuity of the magnetization density at $h = 0$:

$$\lim_{h\downarrow 0}\{m(\beta,h) - m(\beta,-h)\} = 2m^*(\beta) > 0$$

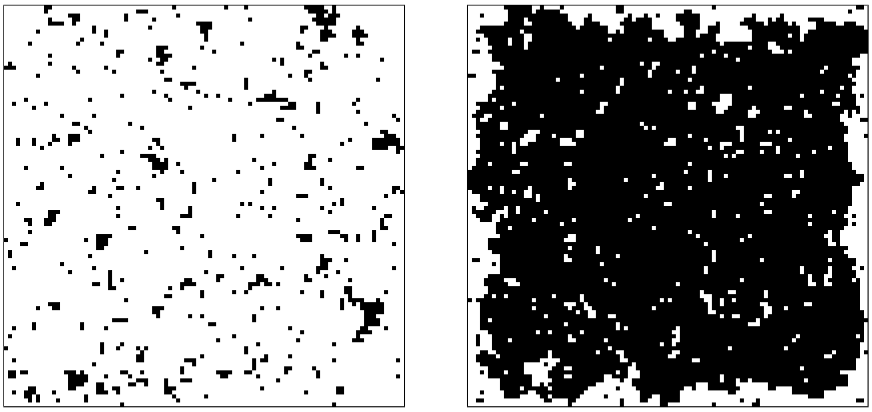


Figure 3.21: Typical low-temperature configurations of the two-dimensional Ising model in a box of sidelength 100 with − boundary condition and magnetic field $h > 0$. Left: For small positive values of h (depending on β and the size of the box), typical configurations are small perturbations of the ground state η^-, even though the − phase is thermodynamically unstable, the cost of creating a droplet of + phase being too large. Right: For larger values of h, the + phase invades the box, while the unstable − phase is restricted to a layer along the boundary, where it is stabilized by the boundary condition. Partial information on the size of this layer (in a slightly different geometrical setting) can be found in [346].

for all $\beta > \beta_c(d)$. It can be shown [352, 7, 8] (for $d = 2$, remember (1.51) and Figure 1.10) that $\beta \mapsto m^*(\beta)$ is decreasing and vanishes continuously as $\beta \downarrow \beta_c$. Therefore, since $m^*(\beta) = 0$ for all $\beta \le \beta_c(d)$, the magnetization density $m(\beta, h)$ (and thus the pressure) cannot be analytic at the point $(\beta_c(d), 0)$. The corresponding phase transition, however, is not of first order anymore: it is said to be **continuous** and the point $(\beta_c(d), 0)$ is said to be a **critical point**.

As we had already mentioned in Section 2.5.3, the behavior of a system at a critical point displays remarkable features. In particular, many quantities of interest have singular behavior, whose qualitative features depend only on rough properties of the model, such as its spatial dimensionality, its symmetries and the short- or long-range nature of its interactions. Models can then be distributed into large families with the same critical behavior, known as **universality classes**.

Among the characteristic features that are used to determine the universality class to which a model belongs, an important role is played by the critical exponents. The definitions of several of the latter have been given for the Curie–Weiss model in Section 2.5.3 and can be used also for the Ising model (using the corresponding quantities). For the Ising model, we have gathered these exponents in Table 3.1:

	$d=2$	$d=3$	$d\geq 4$
α	0	0.110(1)	0
b	1/8	0.3265(3)	1/2
γ	7/4	1.2372(5)	1
δ	15	4.789(2)	3

Table 3.1: Some critical exponents of the Ising model. The exponents given are rigorously only known to hold when $d = 2$ [259, 352, 59] and when $d \geq 4$ [318, 4, 9, 7]. The values given for $d = 3$ are taken from the review [267]; much more precise estimates are now available [194].

Observe that the exponents become independent of the dimension as soon as $d \geq 4$. The dimension $d_{\mathrm{u}} \stackrel{\text{def}}{=} 4$, is known as the **upper critical dimension**. Above d_{u}, the exponents take the same values as in the Curie–Weiss model (see Section 2.5.3), in line with the interpretation of the mean-field approximation as the limit of the model as $d \to \infty$ (see Section 2.5.4). Such a behavior is expected to be general, but with a value of d_{u} depending on the universality class.

At a heuristic level, the core reason for this universality can be traced back to the divergence of the **correlation length** at the critical point. The latter measures the range over which spins are strongly correlated. In the Ising model, the correlation length ξ is such that

$$\langle \sigma_0 ; \sigma_i \rangle^+_{\beta,h} \stackrel{\text{def}}{=} \langle \sigma_0 \sigma_i \rangle^+_{\beta,h} - \langle \sigma_0 \rangle^+_{\beta,h} \langle \sigma_i \rangle^+_{\beta,h} \sim e^{-\|i\|_2/\xi},$$

for all i for which $\|i\|_2$ is large enough. More precisely,

$$\xi(\beta,h)(\mathbf{n}) \stackrel{\text{def}}{=} \lim_{k\to\infty} \frac{-k}{\log\langle \sigma_0 ; \sigma_{[k\mathbf{n}]}\rangle^+_{\beta,h}},$$

where $\mathbf{n}$ is a unit-vector in $\mathbb{R}^d$ and we have written $[x] \stackrel{\text{def}}{=} (\lfloor x(1) \rfloor, \dots, \lfloor x(d) \rfloor)$ for any $x = (x(1), \dots, x(d)) \in \mathbb{R}^d$.

In the Ising model, it is expected that the correlation length is finite (in all directions) for all $(\beta, h) \neq (\beta_c(d), 0)$. This has been proved when $d = 2$ [239]; in higher dimensions, this is only known when either $\beta < \beta_c(d)$ [5] or when β is large enough (we will prove it in Theorem 5.16), while it is known to diverge as $\beta \uparrow \beta_c(d)$ [238].

Under the assumption that there is only one relevant length scale close to the critical point, the divergence of the correlation length implies the absence of any characteristic length scale at the critical point: at this point, the system is expected to be invariant under a change of scale. Based on such ideas, physicists have developed a non-rigorous, but powerful framework in which this picture can be substantiated and which allows the approximate determination of the critical behavior: the **renormalization group**.

Let us briefly describe the idea in a simple case. We define a mapping $T : \Omega \to \Omega$ as follows: given $\omega \in \Omega$, $\omega' \stackrel{\text{def}}{=} T(\omega)$ is defined by

$$\omega'_i \stackrel{\text{def}}{=} \begin{cases} +1 & \text{if } \sum_{j \in 3i + \mathrm{B}(1)} \omega_j > 0, \\ -1 & \text{if } \sum_{j \in 3i + \mathrm{B}(1)} \omega_j < 0. \end{cases}$$

In other words, we partition $\mathbb{Z}^d$ into cubic blocks of sidelength 3, and replace the 3^d spins in each of these blocks by a single spin, equal to $+1$ if the magnetization in the block is positive, and to −1 otherwise. This transformation is called a **majority transformation**.

One can then iterate this transformation. Figure 3.22 shows the first two iterations starting from 3 different initial configuration, corresponding to the two-dimensional Ising model at $h = 0$ and at three values of β: slightly subcritical ($\beta < \beta_c(2)$), critical ($\beta = \beta_c(2)$) and slightly supercritical ($\beta > \beta_c(2)$). At first sight, it looks as though the transformation corresponds to decreasing β in the first case, keeping it critical in the second and increasing it in the third. Of course, the situation cannot be that simple: the probability distribution describing the transformed configuration clearly does *not* correspond to an Ising model anymore. Nevertheless, it might correspond to a model with additional interactions. One could then consider the action of this transformation in the space of all Hamiltonians. The idea is then the following: this transformation has two stable fixed points corresponding to infinite and zero temperatures, which attracts all initial states with $\beta < \beta_c(d)$, respectively $\beta > \beta_c(d)$. In addition it has an unstable fixed point corresponding to the critical point. This can, heuristically, be understood in terms of the correlation length: since each application of the transformation corresponds roughly to a zoom by a factor 3, the correlation length is divided by 3 at each step. As the number of iterations grows, the correlation length converges to 0, which corresponds to $\beta = 0$ or $\beta = \infty$, *except* if it was initially equal to infinity, in which case it remains infinite; this case corresponds to the critical point. An analysis of the behavior of the transformation close to the unstable fixed point then provides information on the critical behavior of the original system.

These ideas are compelling but, at least in this naive form, the above procedure is known to be problematic from a mathematical point of view; see [343] for a detailed discussion or the comments in Section 6.14.2. Nevertheless, more sophisticated versions do allow physicists to obtain remarkably accurate estimates of critical exponents. Moreover, the *philosophy* of the renormalization group has played a key role in several rigorous investigations (even outside the realm of critical phenomena).

From a rigorous point of view, the analysis of critical systems is usually done using alternative approaches, limited to rather specific classes of models and mostly in two situations: systems above their upper critical dimensions and two-dimensional systems. Since research in these fields is still very actively developing, we will not discuss them any further. Instead, we list several good sources where these topics are discussed at length; these should be quite accessible if the reader is familiar with the content of the present book.

A first approach to critical phenomena in lattice spin systems and (Euclidean) quantum field theory, based on random walk (or random surfaces) representations, is exposed in considerable detail in the monograph [102] by Fernández, Fröhlich and Sokal; it provides a thorough discussion of scaling limits, inequalities for critical exponents, the validity of mean-field exponents above the upper critical dimension, etc.

A second approach is described in the books by Brydges [57] and Mastropietro [234]. It is based on a rigorous implementation of a version of the renormalization group. These books cover both the perturbative and nonperturbative renormalization group approaches from the functional-integral point of view and cover a broad spectrum of applications.

A third approach is presented in the book [315] by Slade. The latter provides an

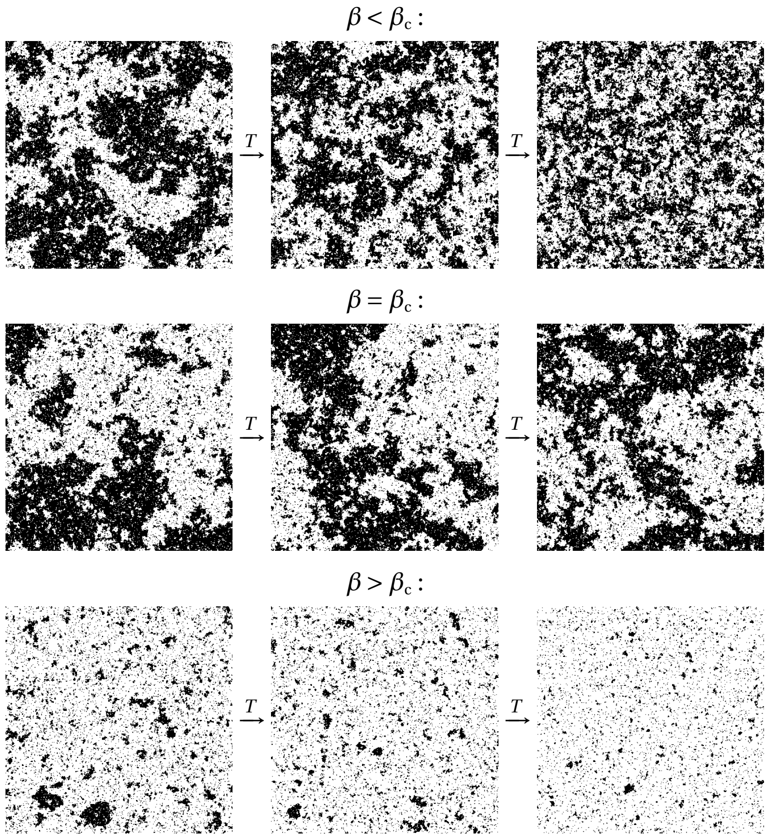


Figure 3.22: Two iterations of the majority transformation at different temperatures.

introduction to the lace expansion, a powerful tool allowing one to obtain precise information on the critical behavior of systems above their upper critical dimension, at least for quantities admitting representation in terms of self-interacting random paths.

A fourth approach, at the base of many of the recent developments of this field, is based on the Schramm–Löwner evolution (SLE). This approach to critical phenomena is restricted to two-dimensional systems, but yields extremely detailed and complete information when it is applicable. An introduction to SLE can be found in the book [210] and in lecture notes by Werner [349] and Lawler [208]. Combined with discrete complex analytic methods and specific graphical representations of spin systems, this approach yields remarkable results, such as the conformal invariance of the scaling limit, explicit expression for the critical exponents, etc. Good references on this topic are the books by Werner [350] and Duminil-Copin [91, 92], as well as the lecture notes by Duminil-Copin and Smirnov [94].

3.10.12 Exact solution

A remarkable feature of the planar Ising model is that many quantities of interest (pressure, correlation functions, magnetization, etc.) can be explicitly computed when $h = 0$. The insights yielded by these computations have had an extremely important impact on the development of the theory of critical phenomena. There exist today many different approaches. The interested reader can find more information on this topic in the books by McCoy and Wu [239], Baxter [17] or Palmer [261], for example.

3.10.13 Stochastic dynamics.

Another topic we have only barely touched upon is the analysis of the stochastic dynamics of lattice spin systems. In the latter, one considers Markov chains on Ω, whose invariant measures are given by the corresponding Gibbs measures. We made use of such a dynamics in Section 3.10.3 in the simplest case of the finite-volume Ising model. The book [225] by Liggett and the lecture notes [232] by Martinelli provide good introductions to this topic.

Revised version, August 22 2017
To be published by Cambridge University Press (2017)

www.unige.ch/math/folks/velenik/smbook