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Linear Glauber model

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We study the time-dependent and the stationary properties of the linear Glauber model in a d-dimensional hypercubic lattice. This model is equivalent to the voter model with noise. By using the Green function method, we get exact results for the two-point correlations from which the critical behavior is obtained. For vanishing noise the model becomes critical with exponents $\beta=0$, $\gamma=1$, and $\nu=1/2$ for $d\geqslant 2$, with logarithmic corrections at the upper critical dimension $d_c=2$, and $\beta=0$, $\gamma=1/2$, and $\nu=1/2$ for $d=1$. We show that the model can be mapped into a particular reaction-diffusion model.

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I. INTRODUCTION

The stochastic process introduced by Glauber [1] assigned to the one-dimensional Ising model a dynamics that the model lacked due to its static definition. The dynamics is a one-spin-flip Markovian stochastic process governed by a master equation whose stationary state is the Gibbs probability associated with the Ising Hamiltonian. This property is ensured by using transition rates that obey detailed balance. The use of a one-spin-flip transition rate guarantees that any spin configuration can be reached from any other so that for large times the spin configurations will be drawn according to the stationary Gibbs probability.

The dynamics introduced by Glauber is not the sole possible dynamics for the Ising model. Glauber himself [1] suggested another one-spin transition rate for the one-dimensional Ising model. Other stochastic dynamics can be set up. One can use, for instance, the Metropolis procedure [2] to construct stochastic processes whose stationary probability is the Gibbs probability associated with the one-dimensional Ising Hamiltonian or any other Ising Hamiltonian in any dimension [3,4]. Indeed, the various Monte Carlo methods available to study the Ising model reflect the great number of such processes [5,6].

Among the various one-spin-flip stochastic processes associated with the Ising model, the original Glauber process is unique in that it can be solved exactly, the basic reason being the linearity of the flip transition rate. More precisely, consider the class of stochastic models described by the one-spin-flip rate w_i associated with the site i of a regular lattice, $w_i(\sigma)=c+\sigma_i g_i(\sigma)$, where σ denotes the collection of Ising spin variable, c is a constant, and $g_i(\sigma)$ is a function of the Ising spin variables. The original Glauber process is such that $g_i(\sigma)$ is linear in the σ variables. If this transition rate is used to describe the dynamics of Ising models in two or more dimensions, the function $g_i(\sigma)$ must be nonlinear.

In this paper we are concerned with the analysis of a stochastic linear model defined in a d-dimensional hypercubic lattice with one-spin-flip transition rate $w_i(\sigma)=c+\sigma_i g_i(\sigma)$ such that $g_i(\sigma)$ is linear in the σ variables. For dimensions greater than 1, such a linear model, which we call the linear Glauber model, does not possess detailed balance so that the stationary probability is not known *a priori*. If, on one hand, the linear Glauber model cannot be the dynamics of the Ising model in two or more dimensions, on the other hand, it can be solved exactly in any dimension and is interesting enough to be studied *per se*.

The linear Glauber model has been the subject of a study by Scheucher and Spohn [7] and can be interpreted as the voter model with noise. In the voter model [8–10] an individual looks at one of its neighbors, chosen at random, and assumes the opinion of this neighbor. With noise, the individual adopts the neighbor's opinion with probability $1-q$ and the opposite opinion with probability q, the noise. The voter model is recovered in the limit of zero noise.

The linear Glauber model (or noise voter model) has been studied in two dimensions by Krapivsky [11] as a model for surface catalytic reaction and by de Oliveira *et al.* [12] and Drouffe and Godrèche [13] in their general study of a class of nonequilibrium models. The voter model is also a special case of models that have been used to study the kinetics of catalytic reactions [11,14], competing learning [15], and critical coarsening without surface tension [16].

For a nonzero noise, the Glauber linear model displays just one phase, the paramagnetic phase. When $q\to 0$, the correlation length diverges and the model becomes critical. Since the voter model displays a nonzero magnetization then, at $q=0$, there is a jump in the magnetization and the model presents a critical first-order transition [12] resulting in a critical exponent $\beta=0$. Other critical exponents of the linear Glauber model are $\nu=1/2$ and $z=2$. The exponent related to the susceptibility is found to be $\gamma=1$ for $d\geqslant 2$, with logarithmic corrections at the upper critical dimension $d_c=2$, and $\gamma=1/2$ for $d=1$.

Dornic *et al.* [16] in their study of coarsening phenomena have defined a class of two-dimensional models for which the ordering occurs with no surface tension. The critical exponents of this class, called the universality class of the voter model [16], are the same as those of the linear Glauber model studied here.

By using the Green function method [17,18] we obtained exact results for the time-dependent as well as the stationary behavior of the magnetization and the two-site correlation function. In particular, we obtained the susceptibility and the density of defects, a measure of the coarsening phenomena, as a function of noise and time. From these results we get the critical behavior and the critical exponents.

II. MODEL

Consider a d-dimensional hypercubic lattice in which at each site $\mathbf{r}$ there is a spin variable $\sigma_{\mathbf{r}}$ that takes the values ± 1. The linear Glauber model is defined by the one-spin-flip rate

$$w_{\mathbf{r}}(\sigma)=\frac{\alpha}{2}\left\{1-\frac{\lambda}{2d}\sigma_{\mathbf{r}}\sum_{\boldsymbol{\delta}}\sigma_{\mathbf{r}+\boldsymbol{\delta}}\right\}, \quad (1)$$

where λ is a parameter in the interval $0<\lambda\leqslant 1$ and the summation is over the $2d$ nearest neighbors of the site $\mathbf{r}$. The parameter α sets the scale of time.

In one dimension, the stationary state of this model has microscopic reversibility and is identified as the Gibbs probability of the one-dimensional nearest-neighbor Ising model. In this case the parameter λ is related to the strength J of the interaction between nearest-neighbor spins and to the temperature T by $\lambda=\tanh(2J/kT)$ [1]. In two or more dimensions there is no such relationship because the model is irreversible, that is, it does not satisfy detailed balance, and we do not know the stationary probability *a priori*.

The model can also be interpreted as a voter model with noise. At each time step a spin is chosen at random. Then one of its $2d$ nearest neighbors is chosen at random. The chosen spin takes the sign of the chosen nearest neighbor with probability $(1+\lambda)/2=1-q$ and the opposite sign with probability $(1-\lambda)/2=q$ which is then interpreted as a noise.

The master equation that governs the time evolution of the probability $P(\sigma,t)$ of configuration $\sigma=\{\sigma_{\mathbf{r}}\}$ at time t is given by

$$\frac{d}{dt}P(\sigma,t)=\sum_{\mathbf{r}}\{w_{\mathbf{r}}(\sigma^{\mathbf{r}})P(\sigma^{\mathbf{r}},t)-w_{\mathbf{r}}(\sigma)P(\sigma,t)\}, \quad (2)$$

where the summation is over the sites of the hypercubic lattice. Here $\sigma^{\mathbf{r}}$ stands for that configuration obtained from σ by changing $\sigma_{\mathbf{r}}$ to $-\sigma_{\mathbf{r}}$.

From the master equation it is straightforward to obtain the time evolution of the magnetization $\langle\sigma_{\mathbf{r}}\rangle$, given by

$$\frac{d}{\alpha dt}\langle\sigma_{\mathbf{r}}\rangle=-\langle\sigma_{\mathbf{r}}\rangle+\frac{\lambda}{2d}\sum_{\boldsymbol{\delta}}\langle\sigma_{\mathbf{r}+\boldsymbol{\delta}}\rangle, \quad (3)$$

and of the pair correlation $\langle\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}\rangle$, $\mathbf{r}'\neq\mathbf{r}$, given by

$$\frac{d}{\alpha dt}\langle\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}\rangle=-2\langle\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}\rangle+\frac{\lambda}{2d}\sum_{\boldsymbol{\delta}}\{\langle\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'+\boldsymbol{\delta}}\rangle+\langle\sigma_{\mathbf{r}'}\sigma_{\mathbf{r}+\boldsymbol{\delta}}\rangle\}, \quad (4)$$

valid for $\mathbf{r}'\neq\mathbf{r}$. We are interested only in the translationally invariant solution of these equations so that $\langle\sigma_{\mathbf{r}}\rangle=m$ will be independent of $\mathbf{r}$ and $\langle\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}\rangle$ will depend only on the difference vector $\mathbf{r}-\mathbf{r}'$, that is, $\langle\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}\rangle=\rho_{\mathbf{r}-\mathbf{r}'}$. The time evolution for m and $\rho_{\mathbf{r}}$ are then given by

$$\frac{d}{\alpha dt}m=-(1-\lambda)m \quad (5)$$

and

$$\frac{1}{\alpha}\frac{d}{dt}\rho_{\mathbf{r}}=-2\rho_{\mathbf{r}}+\frac{\lambda}{d}\sum_{\boldsymbol{\delta}}\rho_{\mathbf{r}+\boldsymbol{\delta}}, \quad (6)$$

which is valid for $\mathbf{r}\neq\mathbf{0}$ with the condition $\rho_0=1$.

Introducing the parameter ε by

$$\varepsilon=\frac{1-\lambda}{\lambda}=\frac{2q}{1-2q}, \quad (7)$$

Eqs. (5) and (6) are written as

$$\frac{d}{dt}m=-\frac{\varepsilon}{2}m \quad (8)$$

and

$$\frac{d}{dt}\rho_{\mathbf{r}}=-\varepsilon\rho_{\mathbf{r}}+\frac{1}{2d}\sum_{\boldsymbol{\delta}}(\rho_{\mathbf{r}+\boldsymbol{\delta}}-\rho_{\mathbf{r}}), \quad (9)$$

where for convenience we have chosen $\alpha=1/2\lambda$. Equation (9) is valid for $\mathbf{r}\neq\mathbf{0}$ with the condition $\rho_0=1$.

The time evolution equations are to be solved for initial states that are translationally invariant. Here we will consider noncorrelated initial states of the Bernoulli type such that $\langle\sigma_{\mathbf{r}}\rangle=m_0$ and $\langle\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}\rangle=m_0^2$. In other words at $t=0$ we have

$$m(0)=m_0 \quad \text{and} \quad \rho_{\mathbf{r}}(0)=m_0^2 \quad (10)$$

except for $\rho_0(0)=1$.

The solution for the magnetization is

$$m(t)=m_0e^{-\varepsilon t/2}, \quad (11)$$

so that $m(\infty)=0$ for $\varepsilon\neq 0$ and $m(\infty)=m_0$ for $\varepsilon=0$. Defining the time correlation length τ by $m=m_0e^{-t/\tau}$ we conclude that it diverges as $\tau\sim\varepsilon^{-\nu_{||}}$ with $\nu_{||}=1$. In the following we will focus on the pair correlation.

III. STATIONARY SOLUTION

A. Pair correlation

In the stationary state we have the equation

$$\frac{1}{2d}\sum_{\boldsymbol{\delta}}(\rho_{\mathbf{r}+\boldsymbol{\delta}}-\rho_{\mathbf{r}})-\varepsilon\rho_{\mathbf{r}}=0, \quad (12)$$

which is valid for $\mathbf{r}\neq 0$ and $\rho_0=1$. In order to solve this equation we start by introducing a parameter a such that

$$\frac{1}{2d}\sum_{\boldsymbol{\delta}}(\rho_{\boldsymbol{\delta}}-\rho_0)-\varepsilon\rho_0=-a. \quad (13)$$

With this condition we can write an equation valid for any site $\mathbf{r}$ of a hypercubic lattice, namely,

$$\frac{1}{2d}\sum_{\boldsymbol{\delta}}(\rho_{\mathbf{r}+\boldsymbol{\delta}}-\rho_{\mathbf{r}})-\varepsilon\rho_{\mathbf{r}}=-a\,\delta_{\mathbf{r}0}, \quad (14)$$

in which the parameter a is to be chosen in such a way that $\rho_0=1$.

The solution of Eq. (14) is obtained as follows. Let $G_{\mathbf{r}}(s)$ be the lattice Green function, defined by

$$G_{\mathbf{r}}(s)=\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{e^{i\mathbf{k}\cdot\mathbf{r}}}{s+d^{-1}\sum_{j=1}^{d}(1-\cos k_j)}\frac{dk_1\cdots dk_d}{(2\pi)^d}, \tag{15}$$

where $\mathbf{r}=(n_1,n_2,\ldots,n_d)$, $n_j=0,\pm1,\pm2,\ldots$ is a site of the d-dimensional hypercubic lattice and $\mathbf{k}=(k_1,k_2,\ldots,k_d)$. It is easily shown, by using a discrete Fourier transform, that $G_{\mathbf{r}}(s)$ is the solution of

$$\frac{1}{2d}\sum_{\boldsymbol{\delta}}(G_{\mathbf{r}+\boldsymbol{\delta}}-G_{\mathbf{r}})-sG_{\mathbf{r}}=-\delta_{\mathbf{r}0}, \tag{16}$$

where the sum is over all nearest neighbors of the origin. Comparing Eqs. (16) and (14), it follows that $\rho_{\mathbf{r}}=aG_{\mathbf{r}}(\varepsilon)$. Since the parameter a should be chosen so that $\rho_0=1$, then $a=1/G_0(\varepsilon)$, and the stationary pair correlation is given by

$$\rho_{\mathbf{r}}=\frac{G_{\mathbf{r}}(\varepsilon)}{G_0(\varepsilon)}. \tag{17}$$

B. Susceptibility

The susceptibility χ is defined as the fluctuation

$$\chi=\sum_{\mathbf{r}}\{\langle\sigma_0\sigma_{\mathbf{r}}\rangle-\langle\sigma_0\rangle\langle\sigma_{\mathbf{r}}\rangle\}. \tag{18}$$

Since in the stationary state $\langle\sigma_{\mathbf{r}}\rangle=0$ for $\varepsilon\neq0$ then

$$\chi=\sum_{\mathbf{r}}\rho_{\mathbf{r}}. \tag{19}$$

Summing each side of Eq. (14) over $\mathbf{r}$ we find the relation $\chi=a/\varepsilon$, which together with the result $a=1/G_0(\varepsilon)$ gives

$$\chi=\frac{1}{\varepsilon G_0(\varepsilon)}. \tag{20}$$

For small values of s, it is well known [18] that $G_0(s)$ is finite for dimensions $d>2$, while in lower dimensions, it diverges when $s\to0$ as

$$G_0(s)=(2s+s^2)^{-1/2}\sim(2s)^{-1/2},\quad d=1, \tag{21}$$

and

$$G_0(s)\sim\pi^{-1}\ln(1/s),\quad d=2. \tag{22}$$

From these results we obtain the asymptotic behavior of the susceptibility $\chi(\varepsilon)$ for small ε as

$$\chi\sim\varepsilon^{-1/2},\quad d=1, \tag{23}$$

and

$$\chi\sim\frac{\varepsilon^{-1}}{|\ln\varepsilon|},\quad d=2. \tag{24}$$

Since $G_0(0)$ is finite for $d>2$ we have

$$\chi=\frac{1}{G_0(0)}\varepsilon^{-1},\quad d>2. \tag{25}$$

Therefore, the susceptibility diverges, in any dimension, as $\chi\sim\varepsilon^{-\gamma}$ with the exponent $\gamma=1/2$ for $d=1$ and $\gamma=1$ for $d\geqslant2$ with logarithmic corrections at the upper critical dimension $d_c=2$.

C. Pair correlation for $\varepsilon=0$

A stationary solution for this case is obtained by taking the limit $\varepsilon\to0$ in Eq. (17). Let us write

$$\rho_{\mathbf{r}}=1-\frac{\tilde{G}_{\mathbf{r}}(\varepsilon)}{G_0(\varepsilon)}, \tag{26}$$

where $\tilde{G}_{\mathbf{r}}(\varepsilon)=G_0(\varepsilon)-G_{\mathbf{r}}(\varepsilon)$ is given by

$$\tilde{G}_{\mathbf{r}}(\varepsilon)=\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{1-\cos\mathbf{k}\cdot\mathbf{r}}{\varepsilon+d^{-1}\sum_{j=1}^{d}(1-\cos k_j)}\frac{dk_1\cdots dk_d}{(2\pi)^d}. \tag{27}$$

In the limit $\varepsilon\to0$, the function $G_0(\varepsilon)$ diverges if $d\leqslant2$, remaining finite if $d>2$. But $\tilde{G}_{\mathbf{r}}(0)$ is a finite quantity in any dimension. Therefore, when $\varepsilon\to0$ we have $\rho_{\mathbf{r}}=1$ if $d\leqslant2$. If $d>2$ we have

$$\rho_{\mathbf{r}}=\frac{G_{\mathbf{r}}(0)}{G_0(0)}, \tag{28}$$

which is strictly less than 1, except for $\mathbf{r}=\mathbf{0}$.

When $s=0$, the divergence of the lattice Green function as defined by Eq. (15), for $d=1$ and $d=2$, means that for $s=0$ no solution of Eq. (16) that decays as $\mathbf{r}\to\infty$ is available. For $d>2$, on the other hand, the Green function is bounded so that Eq. (28) is a solution. Turning back to the original equation (12) for the pair correlation function, we see by inspection that $\rho_{\mathbf{r}}=1$ is a solution of Eq. (12) for any dimension. If $d\leqslant2$ the solution $\rho_{\mathbf{r}}=1$ is the sole solution. However, if $d>2$ this solution is distinct from the solution given by Eq. (28). Actually, in this case any linear combination of these two solutions, such that $\rho_0=1$, will be a solution of Eq. (12). The general stationary solution in three or more dimensions will be then

$$\rho_{\mathbf{r}}=c+(1-c)\frac{G_{\mathbf{r}}(0)}{G_0(0)}, \tag{29}$$

where c is an arbitrary parameter.

As we saw above, the stationary magnetization is m_0 for $\varepsilon=0$. If $m_0\neq\pm1$ there is an apparent contradiction with the stationary solution $\rho_{\mathbf{r}}=1$ for $d\leqslant2$, since $\rho_{\mathbf{r}}=1$ implies a magnetization equal to 1 or -1. This apparent contradiction is resolved by observing that if we start from a configuration with magnetization m_0 then with probability $(1+m_0)/2$ the system will reach magnetization 1 and with probability $(1-m_0)/2$ it will reach magnetization -1.

D. Density of defects

A measure of the coarsening phenomenon is given by the density ρ_{+-} of defects, defined as the density of $+-$ nearest-neighbor pairs of sites. It is related to the pair correlation between two nearest-neighbor sites $\rho_{\mathbf{x}}$, where $\mathbf{x}$ denotes any one nearest neighbor of the origin, that is, $\rho_{+-}=(1-\rho_{\mathbf{x}})/2$. Since $\rho_{\mathbf{x}}=G_{\mathbf{x}}(\varepsilon)/G_0(\varepsilon)$, the density of defects is given by

$$\rho_{+-}=\frac{1}{2}\left\{1-\frac{G_{\mathbf{x}}(\varepsilon)}{G_0(\varepsilon)}\right\}. \tag{30}$$

Now, setting $\mathbf{r}=\mathbf{0}$ in Eq. (16) and using symmetry we obtain a relation between $G_{\mathbf{x}}(s)$ and $G_0(s)$, namely, $G_{\mathbf{x}}(s)-(1+s)G_0(s)=-1$ [20], so that

$$\frac{G_{\mathbf{x}}(s)}{G_0(s)}=1+s-\frac{1}{G_0(s)}. \tag{31}$$

In view of the result (31) and using the asymptotic behavior of the Green function $G_0(s)$ given by Eqs. (21) and (22) we obtain the following results for small values of ε:

$$\rho_{+-}\sim\varepsilon^{1/2}, \quad d=1, \tag{32}$$

$$\rho_{+-}\sim\frac{1}{|\ln\varepsilon|}, \quad d=2, \tag{33}$$

and ρ_{+-} vanishes in the limit $\varepsilon\to 0$. For $d>2$, the density ρ_{+-} approaches a finite value in this limit, namely,

$$\rho_{+-}=\frac{1}{2G_0(0)}, \quad d>2. \tag{34}$$

E. Large $|\mathbf{r}|$ behavior

To obtain the behavior for large values of $|\mathbf{r}|$ we consider the continuum case of Eq. (12), that is, we consider the equation

$$\nabla^2\rho=\alpha^2\rho, \tag{35}$$

where $\alpha=\sqrt{2d\varepsilon}$. Seeking for a spherically symmetric solution we get the equation

$$\frac{d^2\rho}{dr^2}+\frac{d-1}{r}\frac{d\rho}{dr}=\alpha^2\rho, \tag{36}$$

whose solution is of the type

$$\rho=Ar^{-d/2+1}g(\alpha r), \tag{37}$$

where the function $g(x)$ obeys the modified Bessel equation

$$x^2g''(x)+xg'(x)-\left\{x^2+\left(\frac{d}{2}-1\right)^2\right\}g(x)=0, \tag{38}$$

whose solution is the modified Bessel function $K_{d/2-1}(x)$. Therefore the solution is

$$\rho_{\mathbf{r}}=Ar^{-d/2+1}K_{d/2-1}(\alpha r). \tag{39}$$

Using the asymptotic behavior of the Bessel function for large argument we get

$$\rho_{\mathbf{r}}\sim\frac{e^{-\alpha r}}{r^{(d-1)/2}}. \tag{40}$$

Therefore the pair correlation decays exponentially, a result valid for $r\gg\alpha^{-1}$. The correlation length $\xi=\alpha^{-1}$ behaves as $\xi\sim\varepsilon^{-1/2}$ so that $\nu=1/2$ for any dimension. From this result it follows that the exponent $z=\nu_{||}/\nu=2$ for any dimension.

When $\varepsilon=0$ we need to study only the case $d>2$ since for $d\leqslant 2$ we have $\rho_{\mathbf{r}}=1$. If $d>2$ and for large values of $|\mathbf{r}|$ we have to solve

$$\frac{d^2\rho}{dr^2}+\frac{d-1}{r}\frac{d\rho}{dr}=0, \tag{41}$$

to obtain

$$\rho\sim\frac{1}{r^{d-2}}. \tag{42}$$

The pair correlation function then decays algebraically. The exponent η defined through $\rho\sim 1/r^{d-2+\eta}$ is therefore $\eta=0$.

IV. TIME-DEPENDENT SOLUTION FOR $\varepsilon=0$

A. Pair correlation

In this section we will consider only the case $\varepsilon=0$ and the initial condition $m_0=0$. The time evolution of the pair correlation $\rho_{\mathbf{r}}=\langle\sigma_0\sigma_{\mathbf{r}}\rangle$ is given by

$$\frac{d}{dt}\rho_{\mathbf{r}}=\frac{1}{2d}\sum_{\boldsymbol{\delta}}(\rho_{\mathbf{r}+\boldsymbol{\delta}}-\rho_{\mathbf{r}}), \tag{43}$$

valid for $\mathbf{r}\neq\mathbf{0}$ with the condition $\rho_0=1$.

Defining the Laplace transform

$$\hat{\rho}_{\mathbf{r}}(z)=\int_0^\infty\rho_{\mathbf{r}}(t)e^{-zt}dt, \tag{44}$$

we have

$$z\hat{\rho}_{\mathbf{r}}=\frac{1}{2d}\sum_{\boldsymbol{\delta}}(\hat{\rho}_{\mathbf{r}+\boldsymbol{\delta}}-\hat{\rho}_{\mathbf{r}}), \tag{45}$$

valid for $\mathbf{r}\neq\mathbf{0}$, where we have used the initial condition $\rho_{\mathbf{r}}(0)=0$ for $\mathbf{r}\neq\mathbf{0}$. As before we introduce now an equation for $\hat{\rho}_0$, namely,

$$z\hat{\rho}_0=\frac{1}{2d}\sum_{\boldsymbol{\delta}}(\hat{\rho}_{\boldsymbol{\delta}}-\hat{\rho}_0)+\hat{b}, \tag{46}$$

where $\hat{b}(z)$ is to be chosen so that $\hat{\rho}_0=1/z$, which is equivalent to choosing $\rho_0=1$.

Following the same procedure used before one obtains the solution

$$\hat{\rho}_{\mathbf{r}}(z)=\hat{b}G_{\mathbf{r}}(z). \tag{47}$$

Since now $\hat{\rho}_0(z)=1/z$ we get $\hat{b}=1/zG_0(z)$, so that

$$\hat{\rho}_{\mathbf{r}}(z)=\frac{G_{\mathbf{r}}(z)}{zG_0(z)}. \tag{48}$$

The limiting value $\rho_{\mathbf{r}}(\infty)$ can be obtained from

$$\rho_{\mathbf{r}}(\infty)=\lim_{z\to 0} z\hat{\rho}_{\mathbf{r}}(z)=\lim_{z\to 0}\frac{G_{\mathbf{r}}(z)}{G_0(z)}. \tag{49}$$

This limit is 1 for $d\leqslant 2$. If $d>2$ it is strictly less than 1, except for $\mathbf{r}=\mathbf{0}$, and is given by

$$\rho_{\mathbf{r}}(\infty)=\frac{G_{\mathbf{r}}(0)}{G_0(0)}. \tag{50}$$

B. Susceptibility

The Laplace transform $\hat{\chi}(z)$ of the susceptibility

$$\chi(t)=\sum_{\mathbf{r}}\rho_{\mathbf{r}}(t) \tag{51}$$

is given by

$$\hat{\chi}(z)=\sum_{\mathbf{r}}\hat{\rho}_{\mathbf{r}}(z)=\frac{\hat{b}}{z}. \tag{52}$$

The last equality follows from Eqs. (45) and (46). Therefore

$$\hat{\chi}(z)=\frac{1}{z^2G_0(z)}. \tag{53}$$

To determine the long time behavior we need the behavior of $\hat{\chi}(z)$ for small values of z. Using asymptotic results for the Green function given by Eqs. (21) and (22), we have

$$\hat{\chi}(z)\sim\frac{1}{z^{3/2}},\quad d=1, \tag{54}$$

$$\hat{\chi}(z)\sim\frac{1}{z^2|\ln z|},\quad d=2, \tag{55}$$

and

$$\hat{\chi}(z)=\frac{1}{G_0(0)}\frac{1}{z^2},\quad d>2, \tag{56}$$

from which follows

$$\chi(t)\sim t^{1/2},\quad d=1, \tag{57}$$

$$\chi(t)\sim\frac{t}{\ln t},\quad d=2, \tag{58}$$

and

$$\chi(t)=\frac{1}{G_0(0)}t,\quad d>2. \tag{59}$$

If we write $\chi(t)\sim t^{\zeta}$ we have $\zeta=1/2$ for $d=1$ and $\zeta=1$ for $d\geqslant 2$ with a logarithmic correction at the upper critical dimension $d_c=2$. This result for ζ is consistent with the relation $\zeta=(d-2\gamma/\nu)/z$ [19].

C. Density of defects

Let us determine the large t behavior of the density of defects ρ_{+-}, related to the pair correlation $\rho_{\mathbf{x}}$ by $\rho_{+-}=(1-\rho_{\mathbf{x}})/2$. Their Laplace transforms are related by

$$\hat{\rho}_{+-}(z)=\frac{1}{2}\left\{\frac{1}{z}-\hat{\rho}_{\mathbf{x}}(z)\right\} \tag{60}$$

so that

$$\hat{\rho}_{+-}(z)=\frac{1}{2z}\left\{1-\frac{G_{\mathbf{x}}(z)}{G_0(z)}\right\}. \tag{61}$$

Using the result (31) together with the asymptotic behavior of $G_0(s)$ for small values of s given by Eqs. (21) and (22) we have

$$\hat{\rho}_{+-}(z)\sim z^{-1/2},\quad d=1, \tag{62}$$

and

$$\hat{\rho}_{+-}(z)\sim\frac{1}{z|\ln z|},\quad d=2, \tag{63}$$

from which we obtain

$$\rho_{+-}(t)\sim t^{-1/2},\quad d=1, \tag{64}$$

and

$$\rho_{+-}(t)\sim\frac{1}{\ln t},\quad d=2, \tag{65}$$

so that $\rho_{+-}(t)$ vanishes in the limit $t\to\infty$ for $d\leqslant 2$. For $d>2$, the density $\rho_{+-}(t)$ approaches a constant value given by

$$\rho_{+-}(\infty)=\frac{1}{2}\frac{1}{G_0(0)}. \tag{66}$$

V. GENERAL TIME-DEPENDENT SOLUTION

A. Pair correlation

The time evolution of the pair correlation $\rho_{\mathbf{r}}=\langle\sigma_0\sigma_{\mathbf{r}}\rangle$ is given by

$$\frac{d}{dt}\rho_{\mathbf{r}}=\frac{1}{2d}\sum_{\boldsymbol{\delta}}(\rho_{\mathbf{r}+\boldsymbol{\delta}}-\rho_{\mathbf{r}})-\varepsilon\rho_{\mathbf{r}}, \tag{67}$$

valid for $\mathbf{r}\neq\mathbf{0}$ with the condition $\rho_0=1$.

Defining the Laplace transform

$$\hat{\rho}_{\mathbf{r}}(z)=\int_0^\infty \rho_{\mathbf{r}}(t)e^{-zt}dt, \tag{68}$$

we have

$$z\hat{\rho}_{\mathbf{r}}-m_0^2=\frac{1}{2d}\sum_{\boldsymbol{\delta}}(\hat{\rho}_{\mathbf{r}+\boldsymbol{\delta}}-\hat{\rho}_{\mathbf{r}})-\varepsilon\hat{\rho}_{\mathbf{r}}, \tag{69}$$

valid for $\mathbf{r}\neq\mathbf{0}$, where we have used the initial condition $\rho_{\mathbf{r}}(0)=m_0^2$ for $\mathbf{r}\neq\mathbf{0}$. Again we introduce an equation for $\hat{\rho}_0$, namely,

$$z\hat{\rho}_0-m_0^2=\frac{1}{2d}\sum_{\boldsymbol{\delta}}(\hat{\rho}_{\boldsymbol{\delta}}-\hat{\rho}_0)-\varepsilon\hat{\rho}_0+\hat{b}, \tag{70}$$

where $\hat{b}(z)$ is to be chosen so that $\hat{\rho}_0=1/z$, which is equivalent to choosing $\rho_0=1$.

Following the same procedure used before, one obtains the solution

$$\hat{\rho}_{\mathbf{r}}=\frac{1}{(z+\varepsilon)}m_0^2+\hat{b}G_{\mathbf{r}}(z+\varepsilon). \tag{71}$$

Since now $\hat{\rho}_0(z)=1/z$ we get

$$\hat{b}=\left(\frac{1}{z}-\frac{1}{(z+\varepsilon)}m_0^2\right)\frac{1}{G_0(z+\varepsilon)}, \tag{72}$$

so that

$$\hat{\rho}_{\mathbf{r}}=\frac{1}{(z+\varepsilon)}m_0^2+\left(\frac{1}{z}-\frac{1}{(z+\varepsilon)}m_0^2\right)\frac{G_{\mathbf{r}}(z+\varepsilon)}{G_0(z+\varepsilon)}. \tag{73}$$

Let $\rho_{\mathbf{r}}^0(t)$ be the solution corresponding to the initial condition $\rho_{\mathbf{r}}(0)=0$ and $\varepsilon=0$, and let $\hat{\rho}_{\mathbf{r}}^0(z)$ be its Laplace transform. Then from this last equation we have

$$\hat{\rho}_{\mathbf{r}}^0(z)=\frac{G_{\mathbf{r}}(z)}{zG_0(z)}, \tag{74}$$

so that

$$\hat{\rho}_{\mathbf{r}}=\frac{1}{(z+\varepsilon)}m_0^2+\left(1-m_0^2+\frac{\varepsilon}{z}\right)\hat{\rho}_{\mathbf{r}}^0(z+\varepsilon). \tag{75}$$

From this equation it is easy to find that

$$\rho_{\mathbf{r}}(t)=m_0^2e^{-\varepsilon t}+(1-m_0^2)\rho_{\mathbf{r}}^0(t)e^{-\varepsilon t}+\varepsilon\int_0^t\rho_{\mathbf{r}}^0(t')e^{-\varepsilon t'}dt'. \tag{76}$$

Therefore, if the solution $\rho_{\mathbf{r}}^0(t)$ is known, the solution for $\varepsilon\neq0$ and $m_0\neq0$ can be obtained from this equation.

B. Susceptibility

The Laplace transform $\hat{\chi}(z)$ of the susceptibility

$$\chi(t)=\sum_{\mathbf{r}}\{\rho_{\mathbf{r}}(t)-m_0^2e^{-\varepsilon t}\} \tag{77}$$

is given by

$$\hat{\chi}(z)=\sum_{\mathbf{r}}\left\{\hat{\rho}_{\mathbf{r}}(z)-\frac{1}{z+\varepsilon}m_0^2\right\}=\frac{\hat{b}}{z+\varepsilon}. \tag{78}$$

The last equality follows from Eqs. (69) and (70). Therefore

$$\hat{\chi}(z)=\left(1-m_0^2+\frac{\varepsilon}{z}\right)\hat{\chi}^0(z+\varepsilon), \tag{79}$$

where

$$\hat{\chi}^0(z)=\frac{1}{z^2G_0(z)} \tag{80}$$

is the Laplace transform of the susceptibility $\chi^0(t)$ corresponding to the case $\varepsilon=0$ and $m_0=0$. Then

$$\chi(t)=(1-m_0^2)e^{-\varepsilon t}\chi^0(t)+\varepsilon\int_0^t e^{-\varepsilon t'}\chi^0(t')dt'. \tag{81}$$

C. The limit $t\rightarrow\infty$

To get the limit $t\rightarrow\infty$ we use the important property of the Laplace transform

$$\lim_{t\to\infty}F(t)=\lim_{z\to0}z\hat{F}(z), \tag{82}$$

valid for a given function $F(t)$ and its Laplace transform $\hat{F}(z)$. Since

$$z\hat{\rho}_{\mathbf{r}}(z)=\frac{z}{(z+\varepsilon)}m_0^2+(1-m_0^2)z\hat{\rho}_{\mathbf{r}}^0(z+\varepsilon)+\varepsilon\hat{\rho}_{\mathbf{r}}^0(z+\varepsilon), \tag{83}$$

we get

$$\rho_{\mathbf{r}}(\infty)=\varepsilon\hat{\rho}_{\mathbf{r}}^0(\varepsilon)=\frac{G_{\mathbf{r}}(\varepsilon)}{G_0(\varepsilon)}, \tag{84}$$

which is independent of the initial condition m_0^2. Therefore, as long as $\varepsilon\neq0$, the stationary solution (17) is approached no matter what the initial condition is.

For the case $\varepsilon=0$, we have

$$z\hat{\rho}_{\mathbf{r}}(z)=m_0^2+(1-m_0^2)z\hat{\rho}_{\mathbf{r}}^0(z) \tag{85}$$

and

$$\rho_{\mathbf{r}}(\infty)=m_0^2+(1-m_0^2)\lim_{z\to 0}\frac{G_{\mathbf{r}}(z)}{G_0(z)} \tag{86}$$

will depend on the initial condition except for $d\leqslant 2$, since in this case the ratio $G_{\mathbf{r}}(z)/G_0(z)\to 1$ as $z\to 0$ so that $\rho_{\mathbf{r}}(\infty)=1$ independent of the initial condition. For $d>2$, the limiting pair correlation is then

$$\rho_{\mathbf{r}}(\infty)=m_0^2+(1-m_0^2)\frac{G_{\mathbf{r}}(0)}{G_0(0)}, \tag{87}$$

which coincides with the stationary pair correlation (29) with the arbitrary constant $c=m_0^2$.

From the relation $\rho_{+-}=(1-\rho_{\mathbf{x}})/2$ and taking into account the relation (31), one gets the following equation for the density of defects, valid for $d>2$:

$$\rho_{+-}(\infty)=\frac{1}{G_0(0)}\rho_{+-}(0), \tag{88}$$

where we used the fact that $\rho_{+-}(0)=(1-m_0^2)/2$. For $d\leqslant 2$, $\rho_{+-}(\infty)=0$.

In the limit $t\to\infty$ the susceptibility is

$$\chi(\infty)=\lim_{z\to 0}z\hat{\chi}(z)=\frac{1}{\varepsilon G_0(\varepsilon)}, \tag{89}$$

which coincides with the stationary susceptibility found before.

VI. EQUIVALENT REACTION-DIFFUSION MODELS

Consider a d-dimensional hypercubic lattice and suppose that each bond connecting two nearest-neighbor sites can be either occupied by a particle or vacant. We attach to the bond connecting sites $\mathbf{r}$ and $\mathbf{r}'$ a variable $\tau_{\mathbf{r},\mathbf{r}'}$ that takes the value $+1$ or -1 according to whether a bond is occupied or empty. The particles diffuse and react according to certain rules to be specified. We are interested to know the reaction-diffusion system that can be mapped into the linear Glauber model defined by Eq. (1). To this end we make the transformation $\tau_{\mathbf{r},\mathbf{r}'}=\sigma_{\mathbf{r}}\sigma_{\mathbf{r}'}$ so that each configuration $\sigma=\{\sigma_{\mathbf{r}}\}$ is mapped into the configuration $\tau=\{\tau_{\mathbf{r},\mathbf{r}'}\}$.

The spin flip $\sigma_{\mathbf{r}}\to-\sigma_{\mathbf{r}}$ will correspond to changing the variables $\tau_{\mathbf{r},\mathbf{r}'}\to-\tau_{\mathbf{r},\mathbf{r}'}$, that is, to change the sign of the variables attached to $2d$ bonds coming out of site $\mathbf{r}$. In other words, we may say that at each time step the $2d$ particles next to the given site, chosen at random, react in such a way that if a bond is occupied it becomes empty, and vice versa. The transition rate $w_i(\tau)$ of such a reaction is

$$w_{\mathbf{r}}(\tau)=\frac{\alpha}{2}\left\{1-\frac{\lambda}{2d}\sum_{\boldsymbol{\delta}}\tau_{\mathbf{r},\mathbf{r}+\boldsymbol{\delta}}\right\}. \tag{90}$$

In one dimension there are four possible reactions: (a) diffusion to the right or to the left, $(1,-1)\to(-1,1)$ or $(-1,1)\to(1,-1)$, with probability 1/2; (b) creation of two particles, $(-1,-1)\to(1,1)$, with probability $(1+\lambda)/2$; and (c) annihilation of two particles, $(1,1)\to(-1,-1)$, with probability $(1-\lambda)/2$. In a square lattice there are 16 possible reactions. Among them we have the creation and annihilation of four particles with probabilities $(1+\lambda)/2$ and $(1-\lambda)/2$, respectively. We remark that the particles are created and annihilated in an even number. We remark that when $\lambda=1$ (zero noise) the full lattice configuration is an absorbing state.

In one dimension the "σ-τ" transformation above is a one-to-one transformation so that each τ configuration is allowed. In two or more dimensions the "σ-τ" transformation is no longer a one-to-one transformation and not all τ configurations are permitted. However, if one starts with one allowed τ configuration the evolution defined by the transition rate (90) will, of course, always give allowed configurations. In this case, we may ask for the stationary density of particles $\rho_{\mathrm{p}}=[\langle\tau_{\mathbf{r},\mathbf{r}+\mathbf{x}}\rangle+1]/2$. According to the "$\sigma$-$\tau$" transformation, this is given by $\rho_{\mathrm{p}}=[\langle\sigma_{\mathbf{r}}\sigma_{\mathbf{r}+\mathbf{x}}\rangle+1]/2$, or, in the case of translational invariance, $\rho_{\mathrm{p}}=(\rho_{\mathbf{x}}+1)/2=1-\rho_{+-}$. The density of vacancies $\rho_{\mathrm{v}}=1-\rho_{\mathrm{p}}$ equals the density of defects $\rho=\rho_{+-}$.

By virtue of the results obtained for the density of defects we draw the following conclusions for the density of vacancies. For nonzero noise, ρ_{v} approaches, exponentially with time, a finite value which for small values of ε behaves as

$$\rho_{\mathrm{v}}\sim\varepsilon^{1/2},\quad d=1, \tag{91}$$

and

$$\rho_{\mathrm{v}}\sim\frac{1}{|\ln\varepsilon|},\quad d=2. \tag{92}$$

For $d>2$ the density vacancies approach a constant value given by

$$\rho_{\mathrm{v}}=\frac{1}{2G_0(0)},\quad d>2. \tag{93}$$

For zero noise, $\varepsilon=0$, the density of vacancies behaves as

$$\rho_{\mathrm{v}}\sim t^{-1/2},\quad d=1, \tag{94}$$

and

$$\rho_{\mathrm{v}}\sim\frac{1}{\ln t},\quad d=2, \tag{95}$$

for long times, vanishing when $t\to\infty$ so that the system reaches the absorbing state, characterized by $\rho_{\mathrm{v}}=0$. For $d>2$, the density of vacancies approaches a nonzero value which, according to Eq. (88), is given by

$$\rho_{\mathrm{v}}(\infty)=\frac{1}{G_0(0)}\rho_{\mathrm{v}}(0). \tag{96}$$

In this case the absorbing state is not reached and the system remains in an active state.

VII. CONCLUSION

We have investigated the linear Glauber model on a d-dimensional hypercubic lattice. The model is one of the simplest nonequilibrium models that can be solved exactly. By the use of the Green function method we have calculated the two-point correlation functions from which we have obtained the critical behavior and the critical exponents. The model belongs to the universality class of the voter model [16], with exponents $\beta=0$, $\nu=1/2$, and $z=2$ for any dimension. The exponent $\gamma=1$ for $d\geqslant 2$, with logarithmic corrections at the upper critical dimension $d_c=2$, and $\gamma=1/2$ for $d=1$. We have also obtained an exact expression for the density of defects ρ_{+-}. We have also mapped the Glauber model into a reaction-diffusion model which for zero noise has the full lattice as an absorbing state. For $d\leqslant 2$ the absorbing state is always reached, whereas for $d>2$ the final state depends on the initial conditions.

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[1] R.J. Glauber, J. Math. Phys. **4**, 294 (1963).
[2] N. Metropolis, A.W. Rosenbluth, M.N. Rosenbluth, A.H. Teller, and E. Teller, J. Chem. Phys. **21**, 1087 (1953).
[3] L.D. Fosdick, Phys. Rev. **116**, 565 (1959).
[4] J.R. Ehrman, L.D. Fosdick, and D.C. Handscomb, J. Math. Phys. **1**, 547 (1960).
[5] *Monte Carlo Methods in Statistical Physics*, edited by K. Binder, 2nd ed. (Springer-Verlag, Berlin, 1986).
[6] K. Binder and D.W. Heermann, *Monte Carlo Simulation in Statistical Physics*, 2nd ed. (Springer-Verlag, Berlin, 1992).
[7] M. Scheucher and H. Spohn, J. Stat. Phys. **53**, 279 (1988).
[8] T.M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).
[9] M. Howard and C. Godrèche, J. Phys. A **31**, L209 (1998).
[10] E. Ben-Naim, L. Frachebourg, and P.L. Krapivsky, Phys. Rev. E **53**, 3078 (1996).
[11] P.L. Krapivsky, Phys. Rev. A **45**, 1067 (1992).
[12] M.J. de Oliveira, J.F.F. Mendes, and M.A. Santos, J. Phys. A **26**, 2317 (1993).
[13] J.-M. Drouffe and C. Godrèche, J. Phys. A **32**, 249 (1999).
[14] L. Frachebourg and P.L. Krapivsky, Phys. Rev. E **53**, R3009 (1996).
[15] A. Mehta and J.M. Luck, Phys. Rev. E **60**, 5218 (1999).
[16] I. Dornic, H. Chaté, J. Chave, and H. Hinrichsen, Phys. Rev. Lett. **87**, 045701 (2001).
[17] S. Katsura, T. Morita, S. Inawashiro, and Y. Abe, J. Math. Phys. **12**, 892 (1971).
[18] C. Itzykson and J.-M. Drouffe, *Statistical Field Theory* (Cambridge University Press, Cambridge, 1989).
[19] T. Tomé and M.J. de Oliveira, Phys. Rev. E **58**, 4242 (1998).
[20] M. Sahimi, B.D. Hughes, L.E. Scriven, and H.T. Davis, J. Chem. Phys. **78**, 6849 (1983).